

## Article

# Monetary and Fiscal Policy in a Cash-in-advance Economy with Quasi-geometric Discounting\*

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### Abstract

In this paper, we analyze monetary and fiscal policies in a dynamic general equilibrium model in which households have a preference of quasi-geometric discounting and face a cash-in-advance constraint. From this policy analysis, we obtain the following two outcomes.

First, when the government can control only the money supply, the Friedman rule is optimal. Second, when the government can also control income tax rates, the Friedman rule may not be optimal.

*Keywords:* Quasi-geometric discounting; Friedman rule

*JEL classification:* E21, E40, E70

## 1 Introduction

The optimal policy in the case where the government can control both monetary and fiscal policy has been variously discussed, such as in Mulligan and Sala-I-Martin (1997) and Chari and Kehoe (1999). However, this point is not studied sufficiently in the case where households have a preference of quasi-geometric (hyperbolic) discounting, as mentioned

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in later this section. In this study, we incorporate monetary and fiscal policy into a dynamic general equilibrium model with quasi-geometric discounting. Concretely, we use Maeda (2018) which incorporates money into Krusell et al. (2002) by assuming that households face a cash-in-advance (CIA) constraint on consumption. Through this extension, we obtain the following two outcomes: First, when the government can control only money supply, the Friedman rule<sup>1</sup> is optimal. In the model, when households have the preference of the present bias, their labor supply is less than the optimal amount. Generally, the government can increase the labor supply by decreasing the inflation. Therefore, the government makes the nominal interest rate be zero to increase the labor supply. Second, when the government controls both money supply and income tax rates, the optimal inflation rate can be positive. As Maeda (2018) shows, over-saving can occur. In this case, if the government taxes capital income, it can improve welfare by reducing the saving rate. This relaxes the government's budget constraint, so the government decreases the labor income tax, causing the labor supply to increase. However, in some cases, households supply too much labor, in which case the government can improve welfare by increasing inflation to reduce the labor supply.

Next, we survey the related literature, including papers that study a dynamic general equilibrium model with hyperbolic discounting and money. Gong and Zhu (2009) show that money is super-neutral, but they do not analyze the implication of the policy and welfare. Boulware et al. (2013) study the model in which individuals accumulate only money and show that inflation is the cost for the economy. Graham and Snower (2008, 2013) study the New Keynesian model with wage stickiness but without capital accumulation. Graham and Snower (2013) analyze the optimal monetary policy when the government taxes labor income, but only with labor income tax as given. They do not analyze the optimal fiscal policy. Also related to time-inconsistency and money, Hiraguchi (2016) studies the monetary search model which introduces temptation and shows that the Friedman rule is not optimal. Hori et al. (2021) study the non-unitary discount model in which the household's discount rate is different between consumption and labor. They show that the Friedman rule is not optimal when the discount rate of the consumption is higher than that of the labor. However, these studies do not assume hyperbolic discounting, which we incorporate into this study.

The remainder of this paper is organized as follows: Section 2 provides our model. Section 3 covers the optimization of households and the equilibrium. Section 4 lays out the analysis of the government's policy. Section 5 concludes the paper.

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<sup>1</sup> Friedman (1969) argued a monetary policy rule where the optimal nominal interest rate is zero

## 2 The Model

In this section, we explain the goods market, the capital and labor markets, the government, and households in this economy.

**The goods market.** In this economy, a good exists that is produced by inputting capital and labor. The production function is a Cobb-Douglas function. We assume that capital is fully depreciated. Therefore, the goods market clearing condition is as follows:

$$A\bar{k}_{t-1}^\alpha \bar{l}_t^{1-\alpha} = \bar{c}_t + \bar{k}_t, \quad (1)$$

where  $\bar{k}_t$  is the capital accumulated by period  $t$ ,  $\bar{l}_t$  is the labor supply,  $\bar{c}_t$  is consumption,  $A > 0$  is the productivity parameter, and  $\alpha \in (0,1)$  is the capital share<sup>2</sup>.  $\bar{k}_t$ ,  $\bar{l}_t$  and  $\bar{c}_t$  are aggregate values in this economy. We also assume that the goods market is perfectly competitive.

**The capital and labor markets.** These markets are also perfectly competitive, implying marginal-product pricing of the capital and labor inputs:

$$r_t = \alpha A \bar{k}_{t-1}^{\alpha-1} \bar{l}_t^{1-\alpha} \quad (2)$$

$$w_t = (1 - \alpha) A \bar{k}_{t-1}^\alpha \bar{l}_t^{-\alpha}, \quad (3)$$

where  $r_t$  is a real rental price for capital, and  $w_t$  is a real wage.

**Government.** The government issues money at a constant growth rate,  $\theta$ . Therefore, we obtain the dynamics of the real money stock as follows:

$$\bar{m}_t = \frac{1 + \theta}{1 + \pi_t} \bar{m}_{t-1}, \quad (4)$$

where  $\bar{m}_t$  is the real money stock and  $\pi_t$  is the inflation rate. Notice that the government's real income from issuing money at period  $t$  is  $\frac{\theta}{1 + \pi_t} \bar{m}_{t-1}$ . The government taxes households' income from capital and labor, so the government's real income from taxation is:

$$\tau_{g,t} = \tau_r r_t \bar{k}_{t-1} + \tau_w w_t \bar{l}_t, \quad (5)$$

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<sup>2</sup> We assume in the following paragraph that the government transfers all of its income to households, so (1) does not include the government's spending.

where  $\tau_{g,t}$  is the government's real income from taxing,  $\tau_r$  is the tax rate of capital income, and  $\tau_w$  is tax rate of labor income. Moreover, we assume that tax rates stay constant for all time. In this economy, the government transfers all of its income to households:

$$\tau_t = \tau_{g,t} + \frac{\theta}{1 + \pi_t} \bar{m}_{t-1} = \tau_r r_t \bar{k}_{t-1} + \tau_w w_t \bar{l}_t + \frac{\theta}{1 + \pi_t} \bar{m}_{t-1}, \quad (6)$$

where  $\tau_t$  is the real transfer to households.

**Households.** There is one unit of household whose size does not change. We assume each household has one unit of time, divided into leisure and labor. We also assume a utility function with consumption and leisure:

$$u(c_t, l_t) = \ln c_t + \mu \ln(1 - l_t), \quad (7)$$

where  $c_t$  is consumption,  $l_t$  is labor supply, and  $\mu \geq 0$  is the parameter of the preference for leisure. In this economy, there are two assets: capital and money. Therefore, the household's real budget constraint is:

$$k_t + m_t = (1 - \tau_r) r_t k_{t-1} + (1 - \tau_w) w_t l_t + \frac{m_{t-1}}{1 + \pi_t} + \tau_t - c_t, \quad (8)$$

where  $k_t$  is capital and  $m_t$  is money held by the household. We suppose that households face the CIA constraint as follows:

$$c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \frac{\theta}{1 + \pi_t} \bar{m}_{t-1}. \quad (9)$$

Households have the preference of quasi-geometric discounting, so their time utility is:

$$U_t = u(c_t, l_t) + \beta \sum_{s=t+1}^{\infty} \delta^{s-t} u(c_s, l_s), \quad (\beta > 0, 0 < \delta < 1) \quad (10)$$

where  $\beta$  is the parameter of the present bias and  $\delta$  is the discount factor.

### 3 Optimization of households and Equilibrium

In this section, we explain the optimization problem of households and the market equilibrium, using the following notations: Subscripts for all of the variables are omitted, such as  $l$ , and “'” has been added to the superscript of the stock variable, which is accumulated in the next period, such as  $k'$ .

Households choose their behavior,  $c$ ,  $l$ ,  $k'$ , and  $m'$ , by assuming that the rental price of capital  $r(\bar{k}, \bar{l})$ , wages  $w(\bar{k}, \bar{l})$ , inflation rate  $\pi(\bar{k}, \bar{m}, \bar{l})$ , the dynamics of aggregate capital holdings  $\bar{k}' = G(\bar{k}, \bar{m})$ , the dynamics of money  $\bar{m}_t = \frac{1+\theta}{1+\pi_t} \bar{m}_{t-1}$ , and the aggregate labor supply  $\bar{l} = J(\bar{k}, \bar{m})$  are given because these variables depend on aggregate capital  $\bar{k}$ , aggregate labor  $\bar{l}$ , and government policies  $\bar{m}$  and  $\theta$ , which households cannot choose. We assume that the household behaves according to given future decision rules  $k' = g(k, m, \bar{k}, \bar{m}, \bar{l})$ ,  $m' = h(k, m, \bar{k}, \bar{m}, \bar{l})$ , and  $l = j(k, m, \bar{k}, \bar{m}, \bar{l})$  and that the CIA constraint is binding. Here, we seek an equilibrium in which  $G$ ,  $g$ ,  $h$ ,  $J$ , and  $j$  are time-invariant. The problem of the current self is:

$$V_0(k, m, \bar{k}, \bar{m}) = \max_{l, k', m'} \left[ \ln \left( \frac{1}{1+\pi} m + \frac{\theta}{1+\pi} \bar{m} \right) + \mu \ln(1-l) + \beta \delta V(k', m', \bar{k}', \bar{m}') \right] \quad (11)$$

$$s.t. \quad (1 - \tau_r) r(\bar{k}, \bar{l}) k + (1 - \tau_w) w(\bar{k}, \bar{l}) l + \tau_{g,t} = k' + m', \quad (12)$$

where  $V(k, m, \bar{k}, \bar{m})$  is the value function after a period. The current self believes that the future self commits to adopting the decision rules  $g$ ,  $h$ , and  $j$  after a period. Therefore,  $V$  is defined by:

$$\begin{aligned} V(k, m, \bar{k}, \bar{m}) &= \sum_{s=t}^{\infty} \delta^{s-t} u(c_s, l_s) \\ &= \ln \left( \frac{1}{1+\pi} m + \frac{\theta}{1+\pi} \bar{m} \right) + \mu \ln(1 - j(k, m, \bar{k}, \bar{m}, \bar{l})) + \delta V(g(k, m, \bar{k}, \bar{m}, \bar{l}), h(k, m, \bar{k}, \bar{m}, \bar{l}), \bar{k}', \bar{m}'). \end{aligned} \quad (13)$$

The definition of equilibrium is as follows:

**Definition 1.** *A recursive competitive equilibrium for this economy consists of decision rules  $g(k, m, \bar{k}, \bar{m}, \bar{l})$ ,  $h(k, m, \bar{k}, \bar{m}, \bar{l})$ , and  $j(k, m, \bar{k}, \bar{m}, \bar{l})$ , a value function  $V(k, m, \bar{k}, \bar{m})$ , pricing functions  $r(\bar{k}, \bar{l})$ ,  $w(\bar{k}, \bar{l})$  and  $\pi(\bar{k}, \bar{m}, \bar{l})$ , law of motion for aggregate capital  $\bar{k}' = G(\bar{k}, \bar{m})$ , the government's policy  $\bar{m}' = \frac{1+\theta}{1+\pi} \bar{m}$ , and aggregate labor supply  $\bar{l} = J(\bar{k}, \bar{m})$  such that:*

1. given  $V(k, m, \bar{k}, \bar{m})$ ,  $G(\bar{k}, \bar{m})$ ,  $J(\bar{k}, \bar{m})$ ,  $\bar{m}' = \frac{1+\theta}{1+\pi} \bar{m}$ , and the taxes ( $\tau_r$  and  $\tau_w$ ), the rules  $g(k, m, \bar{k}, \bar{m}, \bar{l})$ ,  $h(k, m, \bar{k}, \bar{m}, \bar{l})$ , and  $j(k, m, \bar{k}, \bar{m}, \bar{l})$  solve the optimization problem (11);
2. given  $g(k, m, \bar{k}, \bar{m}, \bar{l})$ ,  $h(k, m, \bar{k}, \bar{m}, \bar{l})$ ,  $j(k, m, \bar{k}, \bar{m}, \bar{l})$ ,  $G(\bar{k}, \bar{m})$ ,  $\bar{m}' = \frac{1+\theta}{1+\pi} \bar{m}$ ,  $J(\bar{k}, \bar{m})$ , and the taxes ( $\tau_r$  and  $\tau_w$ ), the function  $V(k, m, \bar{k}, \bar{m})$  satisfies (13);
3. the CIA constraint is binding;
4.  $r(\bar{k}, \bar{l}) = \alpha A \bar{k}^{\alpha-1} \bar{l}^{1-\alpha}$ ,  $w(\bar{k}, \bar{l}) = (1 - \alpha) A \bar{k}^{\alpha} \bar{l}^{-\alpha}$ ; and
5.  $g(\bar{k}, \bar{m}, \bar{k}, \bar{m}, \bar{l}) = G(\bar{k}, \bar{m})$ ,  $h(\bar{k}, \bar{m}, \bar{k}, \bar{m}, \bar{l}) = \bar{m}'$  and  $j(\bar{k}, \bar{m}, \bar{k}, \bar{m}, \bar{l}) = J(\bar{k}, \bar{m})$ .

If we solve the household optimization problem according to this definition<sup>3</sup>, the saving rate,  $s\alpha$ , and labor supply,  $\bar{l}^*$ , in the recursive competitive equilibrium are given by:

$$s\alpha = \frac{\alpha\beta\delta(\delta + \mu)}{\beta\delta + \mu[1 - \delta(1 - \beta)]}(1 - \tau_r), \quad (14)$$

$$\bar{l}^* = \frac{1 - \alpha}{1 - \alpha + \frac{1 + \theta}{\beta\delta(1 - \tau_w)}\mu(1 - s\alpha)}. \quad (15)$$

If the government does not tax, we can obtain  $s\alpha = \frac{\alpha\beta\delta(\delta + \mu)}{\beta\delta + \mu[1 - \delta(1 - \beta)]}$  and  $\bar{l}^* = \frac{1 - \alpha}{1 - \alpha + \frac{1 + \theta}{\beta\delta}\mu(1 - s\alpha)}$ . These are the same as in Maeda (2018), which assumes no tax.

## 4 Policy Analysis

In this section, we discuss the government's policy, starting with only monetary policy. In the second subsection, we consider both monetary and fiscal policy.

### 4.1 Monetary policy

In this subsection, to consider only the monetary policy,  $\tau_r = \tau_w = 0$ . We define the value function  $V_0(k, m, \bar{k}, \bar{m})$  as the welfare function as in Maeda (2018). The study show that the growth rate of money maximizing the welfare is given by:

$$\bar{\theta} = \beta\delta \frac{1 - s^{op}}{1 - s\alpha} - 1. \quad (16)$$

Here, we focus on the steady state to see if the Friedman rule is optimal. We derive the rental price of capital in the steady state. Using (14) and (15), the amount of capital in next period is given by:  $\bar{k}' = s\alpha A \bar{k}^\alpha (\bar{l}^*)^{1 - \alpha}$ . Combining  $\bar{k}' = \bar{k}$  in the steady state and (2), the rental price of capital in the steady state is:

$$r^* = \frac{1}{s}. \quad (17)$$

The inflation rate in the steady state is equal to  $\theta$ . From (17) and  $\pi = \theta$ , the nominal interest rate in the steady state is:

$$i = \frac{1 + \theta}{s} - 1 \quad (18)$$

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<sup>3</sup> We show this calculation in Appendix A.

Substituting (16) into (18), we obtain the following proposition:

**Proposition 1.** *If  $\beta \leq (>)$  1 in the steady state, then the Friedman rule is (not) optimal.*

*Proof.* Substituting (16) into (18), we obtain the nominal interest rate:

$$\bar{i} = \frac{(\beta-1)\{(1-\alpha\delta)(1+\mu)[\mu(\delta-1)^2+\beta^2\delta^2(1+\mu)]+\beta(1-\delta)\delta[(1-\alpha\delta)((1-\alpha\delta)+\mu(3-\alpha)+\mu^2)+\alpha^2(\delta+\mu)\mu]\}}{\beta(1-\alpha\delta)[1-\delta(1-\beta)](\delta+\mu)[\mu(1-\delta)\beta\delta(1+\mu)]}. \quad (19)$$

If  $\beta \leq (>)$  1, then (19) is negative or 0 (positive). Because the nominal return of money is 0, the nominal interest rate does not negative. Therefore, if  $\beta \leq (>)$  1, the Friedman rule is(not) optimal.  $\square$

This proposition implies that if households are impatient ( $\beta$  is small), then the optimal nominal interest rate is small. Because  $\frac{\partial s\alpha}{\partial \beta} = \frac{\alpha\delta(\delta+\mu)\mu(1-\delta)}{\{\beta\delta+\mu[1-\delta(1-\beta)]\}^2} > 0$ , we find that:

$$\frac{\partial \bar{i}^*}{\partial \beta} = \frac{(1-\alpha) \left[ \delta + \mu(1+\theta) \frac{\partial s\alpha}{\partial \beta} \right]}{[1-\alpha + \frac{1+\theta}{\mu}(1-s)]^2} > 0. \quad (20)$$

This implies that if  $\beta$  is small, then the labor supply may be too small. Therefore, the government decreases the money supply to increase the labor supply. This is the intuitive conclusion that the Friedman rule is optimal.

## 4.2 Monetary and Fiscal policy

Maeda(2018) shows that the optimal saving rate and labor supply that maximize  $V_0(k, m, \bar{k}, \bar{m})$  are given by:

$$s^{op} = \frac{\alpha\beta\delta}{(1-\alpha\delta)[1-\delta(1-\beta)] + \alpha\beta\delta}, \quad (21)$$

$$\bar{l}^{op} = \frac{1-\alpha}{1-\alpha + \mu(1-s^{op})}. \quad (22)$$

Comparing the saving rate in the recursive competitive equilibrium with no tax, (14) and the optimal saving rate, (21), we find that these saving rates are different. However, monetary policy does not affect the saving rate. Therefore, in this subsection, we introduce taxes as well as monetary policy, and seek the optimal policy.

In this economy, the government's transfer does not affect the saving rate, and labor supply. Therefore, we give the amount of transfer exogenously as follows:

$$\tau_{g,t} = 0, \quad (23)$$

$$\tau_t = \frac{\theta}{1+\pi_t} \bar{m}_{t-1} \text{ for all } t. \quad (24)$$

We define  $V_0(k, m, \bar{k}, \bar{m})$  as the welfare function. We can maximize welfare by adjusting the saving rate and labor supply in competitive equilibrium as follows:  $s\alpha = s^{op}$ ,  $\bar{l}^* = \bar{l}^{op}$ . Consequently, we obtain the optimal policy as follows:

$$\tau_r^* = \frac{(1-\beta)\delta(1-\delta)(1-\alpha\delta-\alpha\mu)}{\{(1-\alpha\delta)[1-\delta(1-\beta)]+\alpha\beta\delta\}(\delta+\mu)} \quad (25)$$

$$\tau_w^* = -\frac{\alpha}{1-\alpha} \frac{(1-\beta)\delta(1-\delta)(1-\alpha\delta-\alpha\mu)}{\{(1-\alpha\delta)[1-\delta(1-\beta)]+\alpha\beta\delta\}(\delta+\mu)} \quad (26)$$

$$\theta^* = \beta\delta \left\{ 1 + \frac{\alpha}{1-\alpha} \frac{(1-\beta)\delta(1-\delta)(1-\alpha\delta-\alpha\mu)}{\{(1-\alpha\delta)[1-\delta(1-\beta)]+\alpha\beta\delta\}(\delta+\mu)} \right\} - 1. \quad (27)$$

From (25), (26), and (27), we obtain following proposition.

**Proposition 2.** *In the steady state, when the government uses the monetary and fiscal policy to improve the welfare, the Friedman rule is not optimal if  $\beta < 1$  and  $\mu < \frac{\alpha(1+\delta^2)-\delta(1+\alpha)}{1-\alpha\delta}$ .*

*Proof.* When the government adopts a policy like (25), (26), and (27), the steady-state nominal interest rate is:  $i^* = \alpha \frac{1+\theta^*}{s^{op}} - 1$ . From this equation, we find that if  $1+\theta^* - \frac{s^{op}}{\alpha} > 0$ , then the Friedman rule is not optimal. When we calculate it, we obtain as follows:

$$1+\theta^* - \frac{s^{op}}{\alpha} = \frac{(\beta-1)\beta\delta^2[\delta(1+\alpha)-\alpha(1+\delta^2)+\mu(1-\alpha\delta)]}{\{(1-\alpha\delta)[1-\delta(1-\beta)]+\alpha\beta\delta\}(\delta+\mu)}. \quad (28)$$

This equation implies that if  $\beta < 1$  and  $\mu < \frac{\alpha(1+\delta^2)-\delta(1+\alpha)}{1-\alpha\delta}$ , then  $1+\theta^* - \frac{s^{op}}{\alpha} > 0$ . Therefore, in this case, we find that the Friedman rule is not optimal.  $\square$

The reason why the Friedman rule is not optimal is that over-saving occurs. In this economy, over-saving occurs when  $\mu < \frac{1-\alpha\delta}{\alpha}$  because  $s^{op} - s\alpha = \frac{\alpha\beta\delta^2(1-\delta)(1-\beta)[\alpha\mu-(1-\alpha\delta)]}{\{(1-\alpha\delta)[1-\delta(1-\beta)]+\alpha\beta\delta\}\{\beta\delta+\mu[1-\delta(1-\beta)]\}}$ . Moreover, since  $\frac{\alpha(1+\delta^2)-\delta(1+\alpha)}{1-\alpha\delta} < \frac{1-\alpha\delta}{\alpha}$ , if the Friedman rule is not optimal, then over-saving occurs. When over-saving occurs, the government taxes the capital income to decrease the saving rate. In such a case, the labor income tax is negative because we assume the government's budget constraint is (23). Then households supply labor excessively. This effect is too strong when  $\mu < \frac{\alpha(1+\delta^2)-\delta(1+\alpha)}{1-\alpha\delta}$ . Therefore, the government induces the nominal interest rate to be positive to reduce labor supply. This result is caused by taking distortionary taxes. Our model, in which the government's purpose for imposing distortionary taxes is not only financing its expenditure but also correcting resource allocation, differs from the Phelps (1973) model, in which the government only has to finance its expenditures. Changing resource allocation is the reason that the optimal nominal interest rate is positive.

## 5 Conclusion

In this paper, we covered a general equilibrium model where households have a preference of quasi-geometric discounting and face a cash-in-advance constraint.

There are two contributions from this study. First, we showed that when the government can control the only money supply, the Friedman rule is optimal because households do not supply labor sufficiently when they have the preference of quasi-geometric discounting. Households who have the preference of the present bias increase their leisure and decrease their labor in current period. Therefore, the government decreases the nominal interest rate to increase the labor supply. Second, we show that when the government can control both money supply and income tax rates, there exists the case in which the Friedman rule is not optimal. In our model, oversaving occurs when households have a preference of quasi-geometric discounting and a weak preference for leisure. In such a case, the optimal fiscal policy is reducing the investment in capital by taxing capital income. The government decreases the labor income tax because they have to satisfy their budget constraint. Households supply too much labor, so the government increases the nominal interest rate to a positive level to suppress the labor supply.

## Appendix

### A Derivation of (14) and (15)

We solve this problem by the Guess and Verify method. The value function  $V(k, m, \bar{k}, \bar{m})$  is guessed as follows:

$$V(k, m, \bar{k}, \bar{m}) = B + \ln(m + \theta \bar{m}) - \ln \bar{m} + D \ln \bar{k} + E \ln(k + F \bar{k}), \quad (\text{A1})$$

where B, D, E, and F are constant. Because we assume that (9) is binding, we obtain the first-order conditions with respect to  $k'$ ,  $m'$ , and  $l$  as follows:

$$\frac{\beta \delta E}{k' + F \bar{k}'} - \lambda = 0 \quad (\text{A2})$$

$$\frac{\beta \delta}{m' + \theta \bar{m}'} - \lambda = 0 \quad (\text{A3})$$

$$-\frac{\mu}{1-l} + \lambda w(1 - \tau_w) = 0 \quad (\text{A4})$$

where  $\lambda$  is the Lagrangian multiplier for the budget constraint (12). From (A2), (A3), and (A4), we obtain:

$$k' = E(m' + \theta \bar{m}') - F\bar{k}' \quad (\text{A5})$$

This equation substitutes the budget constraint (12), and we obtain:

$$m' = \frac{1}{1+E} \left\{ (1-\tau_r)rk + (1-\tau_w)wl + \tau_g + F\bar{k}' - E\theta\bar{m}' \right\} \quad (\text{A6})$$

In the equilibrium,  $\bar{c} = \frac{1+\theta}{1+\pi}\bar{m}$  because the CIA constraint (9) is binding. The right-hand side of this equation equals the right-hand side of equation (4). Therefore,  $\bar{c} = \bar{m}'$ . By assumption,  $m = \bar{m}$  and  $k = \bar{k}$ . From these equations, the goods market clearing condition (1), rental price of capital (2), wage (3), and the government's income from taxes (5), we obtain:

$$\bar{m}' = \frac{(1+F)}{(1+F)+E(1+\theta)} A\bar{k}^\alpha \bar{l}^{1-\alpha}. \quad (\text{A7})$$

Moreover, from (A7) and (4), we obtain:

$$\frac{1}{1+\pi} = \frac{(1+F)}{[(1+F)+E(1+\theta)](1+\theta)} \frac{A\bar{k}^\alpha \bar{l}^{1-\alpha}}{\bar{m}}. \quad (\text{A8})$$

Because  $\bar{k}' = A\bar{k}^\alpha \bar{l}^{1-\alpha} - \bar{c} = A\bar{k}^\alpha \bar{l}^{1-\alpha} - \bar{m}'$  from (1) and (9), we obtain:

$$\bar{k}' = G(\bar{k}, \bar{m}) = s\alpha A\bar{k}^\alpha \bar{l}^{1-\alpha} \quad (\text{A9})$$

$$\text{where } s\alpha \equiv \frac{E(1+\theta)}{1+F+E(1+\theta)}. \quad (\text{A10})$$

From (A3) and (A4), we have:

$$1-l = \frac{\mu}{\beta\delta(1-\tau_w)w} (m' + \theta\bar{m}') \quad (\text{A11})$$

From (A6), (A7), and (A9), we obtain:

$$\begin{aligned} & \bar{m} + \theta\bar{m}' \\ &= \frac{1}{1+E} \left\{ (1-\tau_r)rk + (1-\tau_w)wl + \left[ \alpha\tau_r + (1-\alpha)\tau_w + \frac{FE(1+\theta) + \theta(1+F)}{(1+F)+E(1+\theta)} \right] A\bar{k}^\alpha \bar{l}^{1-\alpha} \right\} \end{aligned} \quad (\text{A12})$$

From this equation and (A11), we obtain:

$$\begin{aligned}
 & (1 - \tau_w)wl \\
 &= \frac{1}{\beta\delta(1+E) + \mu} \left\{ \beta\delta(1+E)(1 - \tau_w)w - \mu \left[ (1 - \tau_r)rk + \frac{FE(1+\theta) + \theta(1+F)}{(1+F) + E(1+\theta)} \right] A\bar{k}^\alpha \bar{l}^{1-\alpha} \right\}. \quad (\text{A13})
 \end{aligned}$$

Substituting (2) and (3) into this equation, we obtain:

$$\bar{l}^* = \frac{\beta\delta(1-\alpha)(1-\tau_w)[(1+F) + E(1+\theta)]}{\beta\delta(1-\alpha)(1-\tau_w)[(1+F) + E(1+\theta)] + \mu(1+\theta)(1+F)}. \quad (\text{A14})$$

Moreover, substituting (A13) and (A14) into (A12), we obtain:

$$\begin{aligned}
 & \bar{m} + \theta \bar{m}' \\
 &= \frac{\beta\delta(1-\tau_r)r}{\beta\delta(1+E) + \mu} \left\{ k + \left[ \frac{[(1+E)\beta\delta + \mu](1+F)(1+\theta)}{\alpha\beta\delta(1-\tau_r)[(1+F) + E(1+\theta)]} - 1 \right] \bar{k} \right\} \quad (\text{A15})
 \end{aligned}$$

By definition 1, we must satisfy (13). Therefore, the following equation is satisfied:

$$\begin{aligned}
 & B + \ln(m + \theta\bar{m}) - \ln \bar{m} + D \ln \bar{k} + E \ln(k + F\bar{k}) \\
 &= \delta[B + D \ln s\alpha + E \ln E - D \ln] + (1 - \delta) \ln(1 - s\alpha) + [1 + \delta(D + E)] \ln A \\
 &+ \mu[\ln \mu - \ln(1 - \tau_w) - \ln(1 - \alpha)] + \delta(1 + E) \ln \beta\delta + (\mu + \delta + \delta E)[\ln \alpha + \ln(1 - \tau_r)] \\
 &- \ln(1 + \theta) + \{\alpha\mu + (1 - \alpha)[1 + \mu + \delta(D + E)]\} \ln \bar{l} \\
 &+ \ln(m + \theta\bar{m}) - \ln \bar{m} + [\alpha - \delta - (1 - \alpha)(\mu + \delta E) + \alpha\delta D - \alpha\mu] \ln \bar{k} \\
 &+ (\mu + \delta + \delta E) \ln \left\{ k + \left[ \frac{[(1+E)\beta\delta + \mu](1+F)(1+\theta)}{\alpha\beta\delta(1-\tau_r)[(1+F) + E(1+\theta)]} - 1 \right] \bar{k} \right\}. \quad (\text{A16})
 \end{aligned}$$

From this equation, we obtain:

$$E = \mu + \delta + \delta E \rightarrow E = \frac{\mu + \delta}{1 - \delta} \quad (\text{A17})$$

$$D = \alpha - \delta - (1 - \alpha)(\mu + \delta E) + \alpha\delta D - \alpha\mu \rightarrow D = \frac{\alpha(1 - \delta) - (1 - \alpha\delta)(\mu + \delta)}{(1 - \alpha\delta)(1 - \delta)} \quad (\text{A18})$$

$$\begin{aligned}
 F &= \frac{[(1+E)\beta\delta + \mu](1+F)(1+\theta)}{\alpha\beta\delta(1-\tau_r)[(1+F) + E(1+\theta)]} - 1 \\
 \rightarrow F &= \frac{\beta\delta[1 - \alpha\delta(1 - \tau_r)] + \mu\{[1 - \delta(1 - \beta)] - \alpha\beta\delta(1 - \tau_r)\}}{\alpha\beta\delta(1 - \delta)(1 - \tau_r)}(1 + \theta) - 1 \quad (\text{A19})
 \end{aligned}$$

Finally, these equations substitute (A8), (A10) and (A14), and we obtain:

$$s = \frac{\beta\delta(\mu + \delta)}{\beta\delta + \mu[1 - \delta(1 - \beta)]}(1 - \tau_r)$$

$$\bar{l}^* = J(\bar{k}, \bar{m}) = \frac{1 - \alpha}{1 - \alpha + \frac{1+\theta}{\beta\delta(1-\tau_w)}\mu(1 - s\alpha)}.$$

These equations are the same to (14) and (15).

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