W(δ) Weierstrass points

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The aim of this article is to generalize a notion of Weierstrass points. For any divisor δ we introduce $W(\delta)$ Weierstrass points and $M(1/\delta)$ Weierstrass points, and we estimate the numbers of these points. A $W(1)$ Weierstrass point is a Weierstrass point in the ordinary sense, and a $M(1/\delta)$ Weierstrass point corresponds to a D Weierstrass point defined by means of $H^0(C, (\Omega^1)^{\otimes r(r-1)/2}(rD))$ in the algebraic geometry [2].

1. $W(\delta)$ Weierstrass points.

Let R be a closed Riemann surface of genus g and δ a divisor. Then we have the following Riemann-Roch's theorem.

$\dim M(1/\delta) = \dim W(\delta) + \deg\delta - g + 1.$

Put $k = \dim W(\delta)$ and suppose $k \geqq 1$. For a basis $\psi_j = f_j(z)\, dz$ $(1 \leqq j \leqq k)$ of $W(\delta)$ we write

$$W(z) = \begin{vmatrix} f_1(z) & f_2(z) & f_k(z) \\ f_1^{(1)}(z) & f_2^{(1)}(z) & f_k^{(1)}(z) \\ f_1^{(k-1)}(z) & f_2^{(k-1)}(z) & f_k^{(k-1)}(z) \end{vmatrix}.$$

The above Wronskian defines an Abelian differential Φ_δ of order $k(k+1)/2$, which is independent of a choice of a basis. Consequently we obtain

$$\deg(\Phi_\delta) = k(k+1)(g-1).$$

By the Riemann-Roch's theorem one computes that

$$\dim M(1/\delta\ P^{n-1}) = \dim W(\delta\ P^{n-1}) + \deg\delta + (n-1) - g + 1$$

and

$$\dim M(1/\delta\ P^n) = \dim W(\delta\ P^n) + \deg\ \delta + n - g + 1.$$

This insures

$$\dim M(1/\delta\ P^{n-1}) - \dim M(1/\delta\ P^{n}) = \dim W(\delta\ P^{n-1}) - \dim W(\delta\ P^{n}) - 1.$$

It is easily checked that

$$M(1/\delta\ P^{n}) \supset M(1/\delta\ P^{n-1}) \text{ and } W(\delta\ P^{n-1}) \supset W(\delta\ P^{n}).$$

As an easy calculation shows

$$0 \leqq \dim W(\delta\ P^{n-1}) - \dim W(\delta\ P^{n}) \leqq 1.$$

For each Abelian differential ω we have $\deg(\omega) = 2g-2$. It is easy to see that $W(\delta\ p^{m}) = 0$ where $m = 2g - \deg\delta - 1$. This guarantees that there exist k spaces $W(\delta\ p^{n_j})$ $(1 \leqq j \leqq k)$ such that

$$\dim W(\delta\ P^{n_j-1}) = \dim W(\delta\ P^{n_j}) + 1$$

in the sequence

$$W(\delta) \supset W(\delta\ P) \supset \cdots \supset W(\delta\ P^{m}) = 0.$$

We write

$$0 < n_1 < n_2 < \cdots < n_k < 2g - \deg\ \delta$$

and call it a $W(\delta)$ gap sequence at P. It is clear that

$$M(1/\delta\ P^{n_j}) = M(1/\delta\ P^{n_j-1}) \quad (1 \leqq j \leqq k).$$

P is called a $W(\delta)$ Weierstrass point if

$$\{n_1, n_2, \cdots\cdots, n_k\} \neq \{1,\ 2, \cdots\cdots,\ k\}.$$

We can choose a basis $\{\Psi_j\}$ of $W(\delta)$ such that

$$\Psi_j \in W(\delta\ P^{n_j-1}) - W(\delta\ P^{n_j})$$

for each j $(1 \leqq j \leqq k)$.

Let δ_1 and δ_2 be positive divisors such that

$$\delta = \delta_1/\delta_2 \quad (\delta_1 = S_1^{r_1}\ S_2^{r_2} \cdots\cdots S_s^{r_s},\ \ \delta_2 = T_1^{p_1}\ T_2^{p_2} \cdots\cdots T_t^{p_t}).$$

We consider the following three cases.

Case (i) Without loss of generality we may assume that f_j can be expanded as

$$f_j(z) = z^{n_j-1} + \cdots\cdots$$

in a neighborhood of $P \notin (\bigcup_{u=1}^{s}\{S_u\}) \cup (\bigcup_{v=1}^{t}\{T_v\})$.

As an easy calculation shows the Wronskian can be expanded as

$$W(z) = A\ z^{B} + \cdots\cdots,$$

where

$$A=(-1)^{k(k-1)/2}\prod_{i<j}(n_i-n_j)$$

and

$$B=\sum_{j=1}^{k} n_j-k(k+1)/2.$$

Case (ii) In a neighborhood of a point $P=S_u$, f_j can be expanded as

$$f_j(z)=z^{r_u+n_j-1}+\cdots\cdots \quad .$$

By the same process as (i) one computes

$$B=kr_u+\sum_{j=1}^{k}n_j-k(k+1)/2.$$

Case (iii) In a neighborhood of a point $P=T_v$, f_j can be expanded as

$$f_j(z)=z^{-t_v+n_j-1}+\cdots\cdots.$$

A similar argument yields

$$B=-kt_v+\sum_{j=1}^{k}n_j-k(k+1)/2.$$

From the above three cases we can verify

$$(\Phi_\delta)=\prod_{\substack{P\neq S_u\\ P\neq T_v}}P^{\sum n_j-k(k+1)/2}\times\prod_{u=1}^{s}S_u^{kr_u+\sum n_j-k(k+1)2}$$

$$\times\prod_{v=1}^{t}T_v^{-kt_v+\sum n_j-k(k+1)/2}$$

$$=\prod_{P}P^{\sum n_j-k(k+1)2}\times\delta^k.$$

$$\therefore\quad \deg(\Phi_\delta)=\sum_{j=1}^{k}n_j-k(k+1)/2+k\deg\delta.$$

We denote by $W[\delta]$ the divisor

$$\prod_{P}P^{\sum n_j-k(k+1)/2}$$

and call it a $W(\delta)$ Weierstrass divisor. Then we can obtain the following equation.

$$\deg W[\delta] = -k\deg\delta + k(k+1)(g-1).$$

2. Estimates of the number N of $W(\delta)$ Weierstrass point.

From the $W(\delta)$ gap sequence at P :

$$0 < n_1 < n_2 < \cdots < 2g - \deg\delta.$$

we show that

$$\sum_{j=1}^{k} n_j \leqq \sum_{j=1}^{k} (2g - \deg\delta - j)$$

$$= k(2g - \deg\delta) - k(k+1)/2.$$

$$\therefore \quad \sum_{j=1}^{k} n_j - k(k+1)/2 \leqq K(2g - \deg\delta - 1 - k).$$

This implies that

$$\deg W[\delta] \leqq N\, k(2g - \deg\delta - 1 - k).$$

Suppose $g \geqq 1$ and $k \geqq 2$, then the right term of the above inequality is positive. This insures that

$$\frac{-\deg\delta + (k+1)(g-1)}{2g - \deg\delta - 1 - k} \leqq N \leqq -k\deg\delta + k(k+1)(g-1).$$

Moreover in this case we have

$$2 \leqq N,$$

this guarantees the existence of plural $W(\delta)$ Weierstrass points. In particular, if $\delta = 1$ then $k = \deg W(1) = g$, and the above result can be writtin as

$$g+1 \leqq N \leqq g(g+1)(g-1).$$

This inequality is coarser than the Hurwitz inequality:

$$2(g+1) \leqq N \leqq g(g+1)(g-1).$$

Next we consider the following five special cases.

Case (i) When $g=0$, there exists no $W(\delta)$ Weierstrass point. Because, due to the Riemann-Roch's theorem it turns out that

$$0 \leqq \dim M(1/\delta) = k + \deg\delta + 1.$$

$$k \geqq -1 - \deg\delta.$$

On the other hand, a necessary and sufficient condition for $\deg W[\delta] > 0$ is

$$-\deg\ \delta+(\mathrm{k}+1)(0-1)>0.$$

This gives that

$$k<-1-\deg\ \delta,$$

which contradicts the above inequality.

Case (ii) When $g=1$, a necessary and sufficient condition of existence of a $W(\delta)$ Weierstrass point is $\deg \delta\leqq-1$. First we prove the necessity. For any Abelian differential $\omega\in W(\delta)$ it stands that $\deg(\omega)=2g-2=0$. Therfore $\deg \delta>0$ implies $W(\delta)=0$. This shows by the assumption that $\deg\ \delta\leqq 0$. While if $\deg\ \delta=0$ then

$$\deg\ W[\delta]=-k\times 0+k(k+1)(1-1)=0.$$

We turn to the proof of sufficiency. Using

$$0\leqq\dim\ M(1/\delta)=k+\deg\ \delta$$

we have $k\geqq-\deg \delta\geqq 0$, which insures $W(\delta)\neq 0$. Consequently, by the fact that

$$\deg\ W[\delta]=-k\ \deg\ \delta=k(-\deg\ \delta)\geqq 1\times 1$$

we can conclude $W[\delta]\neq 1$.

Case (iii) When $g\geqq 1$, a point T is a $W(1/T)$ Weierstrass point. We shall point out that the number N of $W(1/T)$ Weierstrass points satisfies

$$g\leqq N\leqq g^3.$$

Because, due to $k=\dim\ W(1/T)=\dim\ W(1)=g$ we have

$$\dim\ W[1/T]=-g\times(-1)+g(g+1)(g-1)=g^3.$$

In particular, when $g=1$ T is the only $W(1/T)$ Weierstrass point. On the other hand, if $g\geqq 2$ then $k=2$. This proves the inequality $g\leqq N\leqq g^3$ by means of the preceding estimate of the number N.

Caes (iv) When $k=1$, a necessary and sufficient condition of existence of $\mathrm{W}(\delta)$ Weierstrass point is

$$g-2\leqq\deg\ \delta<2g-2.$$

This can be demonstrated by the following two formulae.

$$0\leqq\dim\ M(1/\delta)=1+\deg\ \delta-g+1.$$

$$0<\deg\ W[\delta]=-\deg\ \delta+2(\mathrm{g}-1).$$

Case (v) For a point S $W(S)$ Weierstrass points have the property

$$\deg\ W[S]=(g-1)(g^2-g-1).$$

3. Estimates of the number N of $W(1/\delta_2)$ Weierstrass points.

In this section we fix a positive divisor δ_2 such that $\deg\ \delta_2 \geqq 2$. Since $M(\delta_2/P)=M(\delta_2)=0$ it is easy to see that $n_1=1$. Put

$$\alpha=\min\{\{1, 2, 3, \cdots\cdots\}-\{n_1, n_2, \cdots\cdots, n_k\}\}$$

then $\alpha \geqq 2$ is not a gap value. Hence there exists a meromorphic function f such that

$$f \in M(\delta_2/P^\alpha)-M(\delta_2/P^{\alpha-1}).$$

If a integer n is not a gap value at P, then there exists a meromorphic function g such that

$$g \in M(\delta_2/P^n)-M(\delta_2/P^{n-1}).$$

This shows that $n+\alpha$ is not a gap value according to the fact that

$$fg \in M(\delta_2^2/P^n P^\alpha) \subset M(\delta_2/P^{n+\alpha}).$$

After the preceding observations the set of gap values at P can be arranged as follows.

$$\begin{cases} 1, & \alpha+1, & \cdots\cdots\cdots, & m_1\alpha+1, \\ 2, & \alpha+2, & \cdots\cdots\cdots, & m_2\alpha+2, \\ \vdots & & & \\ \alpha-1, & \alpha+(\alpha-1), & \cdots\cdots\cdots, & m_{\alpha-1}+(\alpha-1), \end{cases}$$

where

$$\sum_{p=1}^{\alpha-1}(m_p+1)=\sum_{p=1}^{\alpha-1}m_p+\alpha-1=k$$

and

$$m_p\alpha+p \leqq 2g-\deg\ \delta-1 \quad (1 \leqq p \leqq \alpha-1).$$

It follows from this that

$$\sum_{j=1}^{k}n_j=\sum_{p=1}^{\alpha-1}\sum_{q=1}^{m_p}(q\alpha+1)$$

$$=\sum_{p=1}^{\alpha-1}\{\alpha m_p(m_p+1)/2+(m_p+1)p\}$$

$$=\sum_{p=1}^{\alpha-1}\{m_p\alpha+p)^2/2\alpha-p^2/2\alpha+m_p\alpha/2+p\}.$$

Using

$(m_p\alpha+p)^2\leqq(m_p\alpha+p)(2g-\deg\ \delta-1)$

one can compute the following inequalities.

$$\begin{aligned}\sum_{j=1}^{k} n_j &\leqq (4kg-2k-2k\ \deg\delta+2g-\deg\delta+2\alpha k+\alpha\deg\delta-2\alpha g)/4\\ &\quad -(\alpha-1)(\alpha-2)/6\\ &\leqq k(2g-\deg\delta-1)/2+(2g-\deg\delta)/4+\alpha(k-1)/4\\ &\quad -\alpha(2g-\deg\delta-1-k)/4\\ &\leqq\{2k(2g-\deg\delta-1)+2g-\deg\delta+k(k-1)\}/4\\ &=(2k+1)(2g-\deg\delta)/4+k(k-3)/4.\end{aligned}$$

If $\sum_{j=1}^{k} n_j\leqq F(k)$, where $F(k)$ is independent of P, then the preceding discussion enables us to give the estimate

$$\frac{-k\deg\delta+k(k+1)(g-1)}{F(k)-k(k+1)/2}\leqq N\leqq -k\deg\delta+k(k+1)(g-1).$$

In the case when

$F(k)=(2k+1)(2g-\deg\delta)/4+k(k-3)/4$

we can establish the followong inequality due to dim $W(1/\delta_2)$ $=g+\deg\ \delta_2-1$.

$$\frac{4k^2g}{(2k+1)g+(k-1)^2}\leqq N\leqq k^2g.$$

4. $M(1/\delta)$ Weierstrass points.

For brevity of notation we set $r=\dim M(1/\delta)$. For a basis f_j $(1\leqq j\leqq r)$ of $M(1/\delta)$ we can consider the Wronskian along the same line as the case of $W(\delta)$, and obtain an Abelian differential $\Psi\delta$ of order $r(r-1)/2$, which satisfies

$\deg(\Psi\delta)=r(r-1)(g-1)$.

By the Riemann-Roch's theorem one sees that

$0\leqq\dim M(P^{n-1}/\delta)-\dim M(P^n/\delta)\leqq 1$.

Following the fact that any meromorphic function f satisfies $\deg(f)=0$ we get a sequence of $2+\deg\delta$ spaces:

$M(1/\delta)\supset M(P/\delta)\supset\cdots\cdots\supset M(P^{1+\deg\delta}/\delta)=0$.

By direct calculation we can select $n_j(1\leqq j\leqq r)$ such that

$0<n_1<n_2<\cdots\cdots<n_r<2+\deg\delta$, where

$$\dim M(P^{n_j-1}/\delta)=\dim M(P^{n_j}/\delta)+1.$$

We call the above sequence a $M(1/\delta)$ gap value sequence at P. We remark that

$$W(\delta/P^{n_j-1})=W(\delta/P^{n_j}) \qquad (1\leqq j\leqq r).$$

P is said to be a $M(1/\delta)$ Weierstrass point if

$$\{1, 2, \cdots\cdots, r\}\neq\{n_1, n_2, \cdots\cdots, n_r\}.$$

Let $\{f_j\}$ be meromorphic functions such that

$$f_j\in M(P^{n_j-1}/\delta)-M(P^{n_j}/\delta) \qquad (1\leqq j\leqq r)$$

then $\{f_j|1\leqq j\leqq r\}$ is a basis of $M(1/\delta)$.

In a neighborhood of P, $f_j(z)$ has the following expression.

Case (i) $f_j(z)=z^{n_j-1}+\cdots\cdots$

where $P\notin(\bigcup_{u=1}^{s}\{S_u\})\cup(\bigcup_{v=1}^{t}\{T_v\})$.

Case (ii) $f_j(z)=z^{-r_u+n_j-1}+\cdots\cdots$,

where $P=S_u$.

Case (iii) $f_j(z)=z^{t_v+n_j-1}+\cdots\cdots$,

where $P=T_v$.

In much the same as the discussion of $W(\delta)$ Weierstrass points we can prove

$$(\Psi\delta)=\prod_P P^{\Sigma n_j-r(r+1)/2}\cdot\delta^{-r}.$$

Define

$$M[1/\delta]=\prod_P P^{\Sigma n_j-r(r+1)/2}$$

and we call it a $M(1/\delta)$ Weierstrass divisor. It is not difficult to see that

$$\deg M[1/\delta]=r\deg\delta+r(r-1)(g-1).$$

5. Estimates of the number N of $M(1/\delta)$ Weierstrass points.

By the same argument as that used in the estimate of the number of $W(\delta)$ Weierstrass points it is easy to calculate N as follows.

$$\sum_{j=1}^{r} n_j \leqq \sum_{j=1}^{r}(2+\deg\delta-j).$$

$$\therefore \quad \sum_{j=1}^{r} n_j - r(r+1)/2 \leqq r(\deg\delta+1-r).$$

$$\therefore \quad \deg M[1/\delta] \leqq Nr(\deg\delta+1-r).$$

While, if $g\geqq 1$ and $r\geqq 2$, then the right term of the above inequality is positive. So it is easy to check that

$$\frac{\deg\delta+(r-1)(g-1)}{\deg\delta+1-r} \leqq N \leqq r\deg\delta+r(r-1)(g-1).$$

Moreover in this case it is apparent that $N\geqq 2$.

We shall observe the following three special cases.

Case (i) When $g=0$ we see

$$r-\deg\delta-1=\dim W(\delta)\geqq 0.$$

On the other hand it stands that

$$\deg M[1/\delta]=r(\deg\delta-r+1),$$

which implies $\deg M[1/\delta]\leqq 0$, that is, there exists no $M(1/\delta)$ Weierstrass point.

Case (ii) When $g=1$, $\deg\delta\geqq 1$ is a necessary and sufficient condition of the existence of a $M(1/\delta)$ Weierstrass point. First we prove the necessity. Let f be a meromorphic function, then $\deg(f)=0$. Consequently, if $\deg(1/\delta)\geqq 1$, then $M(1/\delta)=0$, which is a contradiction. This proves $\deg\delta\geqq 0$. If $\deg\delta=0$, then $\deg M[1/\delta]=0$, which is impossible. Hence we find that $\deg\delta\geqq 1$. Next we prove the sufficiency. By the fact that

$$r=\dim W(\delta)+\deg\delta\geqq 1$$

we deduce that $\deg W[1/\delta]=r\deg\delta\geqq 1$, which guarantees $W[1/\delta]\neq 1$.

Case (iii) When $g\geqq 1$, a point S is the only $M(1/S)$ Weierstrass point. Because, $\dim M(1/S)=\dim M(1)=1$ shows $r=1$. Thus $\deg\delta=\deg S=1$ proves $\deg M[1/\delta]=1\times 1=1$.

This means that there exists the only $M(1/S)$ Weierstrass point.

While,

$$\text{if } P=S, \text{ then } M(1/S)=M(P/S)=M(1),$$

or

if $P \neq S$, then $M(1/S) \supset M(P/S) = 0$,

Which shows that S is a $M(1/S)$ Weierstrass point.

6. Relations between $W(\delta)$ Weierstrass points and $M(1/\delta)$ Weierstrass points.

Using the three equations

$$r = k + \deg \delta - g + 1,$$

$$\deg W[\delta] = -k \deg \delta + k(k+1)(g-1)$$

and

$$\deg M[1/\delta] = r \deg \delta + r(r-1)(g-1)$$

we are able to prove that

$$2g^2 J^2 (\deg W[\delta] + \deg M[1/\delta])$$
$$= (\deg W[\delta] - \deg M[1/\delta])^2 (g-1) + g^2 J^4 (g+1),$$

where $J = \deg \delta - g + 1$.

In the remainning part we shall consider the case where $g=2$ and $\delta = S$. In this case, $J=0$ insures $\deg W[S] = \deg M[1/S] = 1$. This means that there exists the only $W(S)$ Weierstrass point. Using $M(1/S) = M(1)$ we get $r = \dim M(1) = 1$. Therefore by the Riemann-Roch's theorem one finds $k = \dim W(S) = 1$. Following this, the gap value at P is $n_1 = 1$ or $n_2 = 2$. It follows that $W(S) = W(SP)$ is a necessary and sufficient condition that P is a $W(S)$ Weierstrass point. Note that $W(S) = W(SP)$ yields $M(1/SP) - M(1/S) \neq \phi$. Therefore in order that S is a $W(S)$ Weierstrass point it is necessary and sufficient that there exists a meromorphic function which has the only singularity of order two at S. This means that S is a Weierstrass point in the ordinary sense. Define a mapping G by

$G : S \longrightarrow P$ (P is the $W(S)$ Weierstrass point),

then G is an involution of hyperelliptic Riemann surface.

References

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