ON NON-UNIFORM ESTIMATES OF ASYMPTOTIC EXPANSIONS IN THE CENTRAL LIMIT THEOREM

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The leading term approach to rates of convergence is employed to derive non-uniform and global descriptions of the rate of convergence in the central limit theorem. Both upper and lower bounds are obtained, being of the same of magnitude, modulo terms of order $n^{-r}$. We follow the approach developed in [3], which permits general results to be obtained by considering only those expansions with an odd number of terms.

1. Introduction

Our purpose in this paper is to investigate several non-uniform and global descriptions of Edgeworth-type approximations in the central limit theorem. We apply the leading term approach, developed in [2, 3], and base our results on a new non-uniform bound for the error remaining after approximating by the leading term in a Chebyshev-Edgeworth-Cramér expansion; see Theorem 1. This result, and a new estimate of the leading term itself (Theorem 2), lead to upper bounds for rates of convergence in non-uniform metrics, weighted $L^p$ metrics and unweighted $L^p$ metrics (Corollary 1). Lower bounds are also considered. We derive new lower bounds to rates in the $L^p$ metric,
in which integration is carried out only over an interval containing the origin (Theorem 3). These results are combined (Corollary 2) to complement earlier work by Heyde and Nakata [4], where the case of simple normal approximation was examined.

In all cases, our upper and lower bounds are of the same order of magnitude, modulo terms in \( n^{-1} \). They represent an attempt to determine the "precise" rate of convergence of remainder terms in various metrics.

In this sense, our results are most closely related to asymptotic expansion theory developed in [1, 3]. Petrov [5, Chapter VI] has given an excellent account of the classical theory of approximation by Chevyshev-Edgeworth-Cramér expansions.

We conclude this section with notation. Let \( X, X_1, X_2, \ldots \) be independent and identically distributed random variables with zero mean and unit variance, and define

\[
S_n = \sum_{i=1}^{n} X_i, \quad F_n(x) = P(S_n < nx)
\]

and

\[
\Phi(x) = \int_{-\infty}^{x} e^{u^2/2} \, du, \quad \text{for} \quad -\infty < x < \infty. \quad \text{We assume that}
\]

\[
E(X^{2k+2}) < \infty \quad \text{for an integer} \quad k > 0, \quad \text{and let} \quad \mu_j = E(X^j) \quad \text{for}
\]

\( 1 < j < 2k + 2, \) with \( \mu_{2k+3} \) defined arbitrarily.

The "cumulants" \( \kappa_j, \ j > 1, \) are defined by the formal expression,

\[
\exp \left( \sum_{j=1}^{\infty} \frac{\kappa_j u^j}{j!} \right) = 1 + \sum_{j=1}^{2k+3} \frac{\mu_j u^j}{j!}.
\]

Of course, only \( \kappa_1, \kappa_2, \ldots, \kappa_{2k+2} \) are true cumulants. The quantity \( \kappa_{2k+3} \) is a "pseudocumulant", in that it depends on the arbitrary (but fixed) term \( \mu_{2k+3} \), as well as \( \mu_3, \ldots, \mu_{2k+2} \). Pseudocumulants of order
2k + 4 and higher are not used in the work which follows.

Following Hall [1, 3], we define polynomials $P_j$ and $Q_j$, $j \geq 1$, by

\[
\exp \left( \sum_{j=3}^{\infty} \kappa_j u^{j-2} / j! \right) = 1 + \sum_{j=1}^{\infty} P_j (u) v^j \quad \text{and} \quad \int_{-\infty}^{\infty} e^{itx} \psi (Q_j (x)) = P_j (it) e^{-t^2 / 2},
\]

where $\psi = \Phi'$ and $i = \sqrt{-1}$. Note that the first $2k - 1$ $P_j$'s and $Q_j$'s depend only on the first $2k + 3$ $\kappa_j$'s. Observe too that $P_j (0) = 0$ for all $j$. The leading term function is

\[
_k L_n (x) = n \mathbb{E} \{ \Phi (x - X / n^{1/2}) \Phi (x) - \sum_{j=1}^{2k + 2} (-X / n)^{j - 1} \psi^j (x) / j! \} + \mu_{2k + 3} n^{-(2k + 1)/2} \Phi^{(2k + 2)} (x) / (2k + 3)!,
\]

and is roughly of order of magnitude

\[
_k \delta_n = n^{-k} \mathbb{E} \{ X^{2k + 2} \mathbb{1} (|X| > n^{1/2}) \} + n^{-(k + 1)} \mathbb{E} \{ X^{2k + 4} \mathbb{1} (|X| \leq n^{1/2}) \} + n^{-(2k + 1)/2} \mathbb{E} \{ X^{2k + 2} \mathbb{1} (|X| \leq n^{1/2}) \} - \mu_{2k + 3}.
\]

Indeed, it was shown in [3] that the ratio

\[
\sup_{-\infty < x < \infty} \left| _k L_n (x) \right| / _k \delta_n
\]

is bounded away from zero and infinity as $n \to \infty$.

The function $_k L_n$ represents a first-order approximation to the error term,

\[
F_n (x) - \Phi (x) - \Phi (x) \sum_{j=1}^{2k + 1} Q_j (x) n^{-j/2}.
\]

Thus, we consider only those Chebyshev-Edgeworth-Cramér expansions of the normal error which consist of an odd number of terms. This actually confers more generality than the classical approach. For
example, under the assumption $E(X^{2k+2}) < \infty$, we obtain an expansion of the nominal error in which the remainder term is of order $n^{-(k+1)}$. The usual approach would require the moment condition $E|X|^{2k+3} < \infty$ before achieving a remainder of this order. Thus, we are able to bridge the gap between $n^{-k}$ and $n^{-(k+1)}$ with a single term, $L_n(x)$, under minimal moment conditions. The traditional approach requires the gaps between $n^k$ and $n^{-(2k+1)/2}$, and between $n^{-(2k+1)/2}$ and $n^{-(k+1)}$, to be considered separately. See for example Theorem 1, page 159 of Petrov [5].

2. Results

Our first result is basic to our approach. It presents a non-uniform description of the error in a Chebyshev-Edgeworth-Cramér expansion, after the leading term has been taken into account. A uniform version is given in Theorem 1 of [3].

**THEOREM 1.** Assume $E(X^{2k+2}) < \infty$ for any integer $k \geq 0$, and that Cramér’s condition,

$$\limsup_{t \to \infty} |E(e^{itX})| < 1,$$

(C)

holds. Then

$$\sup_{-\infty < x < \infty} |F_n(x) - \Phi(x) - \Phi(x) \sum_{j=1}^{2k+1} Q_j(x) n^{-j/2} - L_n(x)|$$

$$= O(n^{-k} + (\delta_n^2 + n^{(k+1)}))$$

(2.1)

as $n \to \infty$.

A key aspect of this theorem is that the right hand side of (2.1) equals $o(\delta_n) + O(n^{-(k+1)})$. Our next estimate shows that the leading term is of order $O(\delta_n)$, in the same non-uniform metric.
THEOREM 2. Assume \( \mathbb{E}(X^{2k+2}) < \infty \). There exists a constant \( C > 0 \), depending only on \( k \) such that

\[
\sup_{-\infty < x < \infty} \left( 1 + |x|^{2k+2} \right) |L_n(x)| \leq C_k \delta_n \tag{2.2}
\]

for all \( n \).

Results (2.1) and (2.2) lead immediately to the following corollary, which gives upper bounds to the rate of convergence in non-uniform and weighted \( L^p \) metrics.

COROLLARY 1. Assume \( \mathbb{E}(X^{2k+2}) < \infty \), and that Cramér's condition \((C)\) holds. Then

\[
\sup_{-\infty < x < \infty} \left( 1 + |x|^{2k+2} \right) |F_n(x) - \Phi(x) - \phi(x) \sum_{j=1}^{2k+1} Q_j(x) n^{-1/2} | = O_k(\delta_n + n^{-(k+1)}) \tag{2.3}
\]

as \( n \to \infty \), and for any \( p \geq 1 \) and \( r < p(2k+2)-1 \),

\[
\left\{ \int_{-\infty}^{\infty} \left( 1 + |x|^r \right) |F_n(x) - \Phi(x) - \phi(x) \sum_{j=1}^{2k+1} Q_j(x) n^{-1/2} |^p dx \right\}^{1/p} = O_k(\delta_n + n^{-(k+1)}) \tag{2.4}
\]

as \( n \to \infty \).

In the case of the non-uniform metric, a lower bound may be deduced from a lower bound in the uniform metric. Arguing in this way, we may deduce from Theorem 4 of [3] that the left-hand side of (2.3), plus \( n^{-(k+1)} \), is bounded below by a constant multiple of \( \delta_n \), for all large \( n \). However, we need a little extra theory if we are to deduce a lower bound in the case of the \( L^p \) metric. To this end, we establish the following result.

THEOREM 3. Assume \( \mathbb{E}(X^{2k+2}) < \infty \). For any \( p \geq 1 \) and \( \epsilon > 0 \),
\[
\lim \inf \left( \int_0^\infty \left( 1 + x^{1 - r} \right) |k L_n(x)|^p dx \right)^{1/p} / k \delta_n > 0
\]  
(2.5)

and

\[
\lim \inf \left( \int_{-1}^0 \left( 1 + x^{1 - r} \right) |k L_n(x)|^p dx \right)^{1/p} / k \delta_n > 0.
\]

It follows immediately from Theorems 1 and 3 that for any \( r \), the quantity

\[
\left\{ \int_{-\infty}^\infty (1 + x^{1 - r}) |F_n(x) - \Phi(x) - \phi(x) \sum_{i=1}^{k+1} Q_i(x) n^{i/2} | ^p dx \right\}^{1/p} + n^{-(k+1)},
\]

is bounded below by a constant multiple of \( k \delta_n \) as \( n \to \infty \). We may combine this result with the upper bound at (2.4), as follows. Assume that \( E(X^{2k+2}) < \infty \), but that the tails of \( X \) are sufficiently large for \( x^{2k+4} P(\mid X \mid > x) \to \infty \) as \( x \to \infty \). Then

\[
k \delta_n \geq n^{-k} \int_{n^{1/2}}^{\infty} x^{2k+2} dP(\mid X \mid < x)
\]

\[
\geq n^{-(k+1)} n^{(2k+4)/2} P(\mid X \mid > n^{1/2})
\]

\[
= n^{(k+1)} \lambda_n,
\]
say, where \( \lambda_n \to \infty \) as \( n \to \infty \). In this case, the terms \( n^{(k+1)} \)

appearing at (2.4) and (2.6) are negligible in comparison with \( k \delta_n \).

Thus, we have:

COROLLARY 2. Assume \( E(X^{2k+2}) < \infty \), Cramér's condition (C), and that

\( x^{2k+4} P(\mid X \mid > x) \to \infty \)

as \( x \to \infty \). Then for any \( p \geq 1 \) and \( r < p (2k - 2) - 1 \), the ratio

\[
\left( \int_{-\infty}^\infty (1 + x^{1 - r}) |F_n(x) - \Phi(x) - \phi(x) \sum_{i=1}^{k+1} Q_i(x) n^{i/2} | ^p dx \right)^{1/p} / k \delta_n
\]
is bounded away from zero and infinity as \( n \to \infty \).

Corollary 2 extends a portion of Theorem 1 of Heyde and Nakata [4], where the case \( k = 0 \) was considered.

3. Proofs

PROOF OF THEOREM 1. Throughout the proof, symbols \( C \) and \( p \) denote respectively a generic positive constant and a generic positive integer. Much of the proof consists of deriving bounds for derivatives of the form

\[
(d / dt)^{\ell} f(t), \quad 0 \leq \ell \leq 2k + 2,
\]

for various functions \( f \). The constants \( C \) and \( p \) appearing in these bounds depend on \( \ell \), but since \( \ell \) takes only a finite number of values, \( C \) and \( p \) may in fact be chosen so that the bounds apply uniformly in \( \ell \).

The characteristic function (Fourier-Stieltjes transform) of

\[
k D_n(x) = F_n(x) - \Phi(x) - \sum_{j=1}^{2k+1} P_j(-\Phi(x)) n^{-j/2} - k L_n(x),
\]

is given by

\[
k d_n(t) = a^n(t / n^{1/2}) - e^{-t^2/2} \sum_{j=0}^{2k+1} P_j(it) n^{-j/2} \]
\[
- n [a (t / n^{1/2}) - \sum_{j=0}^{2k+3} \mu_j (it / n^{1/2}) / j!] e^{-t^2/2}.
\]

The Fourier-Stieltjes transform of \( x^{2k+2} k D_n(x) \), equals

\[
\chi(t) = \int_{-\infty}^{\infty} e^{itx} d_n \{ x^{2k+2} k D_n(x) \}
\]
\[
= it (d / dt)^{2k+2} \{ (it)^{-1} k d_n(t) \}.
\]

Consequently,

\[
|\chi(t)| \leq C \sum_{r=0}^{2k+2} t^r \left| k d_n^{(2k+2+r)}(t) \right|.
\]
for all \( t > 0 \). From this result, and a version of the smoothing inequality for Fourier-Stieltjes transforms (see Lemma 8, p. 155 of Petrov [5]), we may deduce that

\[
\sup_{-\infty < x < \infty} \left(1 - \left| x \right|^{2k+2}\right) \left| \int_0^T \left(1 - t^{2k+3}\right) \left| D_n(t) \right| dt + \sum_{r=0}^{2k+1} \int_0^T \left| D_n(t) \right| dt + T^{1}\right)
\]

(3.1)

for all \( T > 1 \), provided

\[
\sup_{n \geq 1} \sup_{-\infty < x < \infty} \left(1 + \left| x \right|^{2k+2}\right) \left| \frac{d}{dx} \{ D_n(x) - F_n(x) \} \right| < \infty .
\]

(3.2)

Condition (3.2) will follow if we prove that

\[
\sup_{n \geq 1} \sup_{-\infty < x < \infty} \left(1 + \left| x \right|^{2k+2}\right) \left| L_n'(x) \right| < \infty ,
\]

or equivalently,

\[
\sup_{n \geq 1} \sup_{-\infty < x < \infty} \left(1 + \left| x \right|^{2k+2}\right) n \left| E \{ \phi(x - X / n^b) - \phi(x) \} \right| < \infty .
\]

(3.3)

Since

\[
\left| E \{ \phi(x - X / n^b) - \phi(x) \} \right| \leq E \left( X / n^b \right)^2 \sup_x \left| \phi''(x) \right| ,
\]

then (3.3) will follow if we show that

\[
\sup_{n \geq 1} \sup_{-\infty < x < \infty} n \left| x \right|^{2k+2} \left| E \{ \phi(x - X / n^b) - \phi(x) \} \right| < \infty .
\]

(3.4)

If \( x \geq 2 \) and \( |X / n^b| > x / 2 \), then

\[
\left| \phi(x - X / n^b) - \phi(x) \right| \leq 1 ;
\]

if \( x \geq 2 \) and \( 1 < |X / n^b| < x / 2 \), then

\[
\left| \phi(x - X / n^b) - \phi(x) \right| < \phi(x / 2) ;
\]

and if \( x \geq 2 \) and \( |X / n^b| = 1 \), then
Therefore

\[ \left| \mathbb{E} \{ \phi(x - X / n^{1/2}) - \phi(x) \} \right| \]

\[ \leq P \left( \left| X \right| > n^{1/2} x / 2 + \phi(x / 2) - P \left( \left| X \right| > n^{1/2} \right) \right. \]

\[ + n^{-1/6} \mathbb{E} \{ \mathbb{X} \left( \left| X \right| \leq n^{1/2} \right) \} \left| \phi' (x) \right| + n^{-1} \sup_{x - 1 \leq y \leq x + 1} \left| \phi'' (y) \right| \]

\[ \leq E \{ X^{2^{k+2}} / n^{1/2} x \}^{2^{k+2}} + n^{-1} \{ \phi(x / 2) + | \phi'(x) | + \sup_{x - 1 \leq y \leq x + 1} \left| \phi''(y) \right| \}. \]

Consequently,

\[ \sup_{n \geq 1} \sup_{x \geq 2} n x^{2^{k+2}} \left| \mathbb{E} \{ \phi(x - X / n^{1/2}) - \phi(x) \} \right| < \infty. \]

The case where \( x \leq -2 \) may be treated similarly. Therefore (3.4) (and thus (3.2)) is proved.

The remainder of the proof consists of estimating the terms on the right hand side of (3.1). We first estimate

\[ d^{(n)}_n (t) = \left( \frac{d}{dt} \right)^k d_n (t), \]

for \( t \) in the range \( 0 < t \leq n^\delta \), and some \( \delta \in (0, 1/2) \). This is carried out in several stages, the first being to estimate

\[ \left( \frac{d}{dt} \right)^k \left[ e^{n (t / n^{1/2})} - \exp(n \{ \alpha(t / n^{1/2}) - 1 - \sum_{j=2}^{2^{k+3}} \mu_j (it / n^{1/2}) / j! \}ight. \]

\[ + \left. \sum_{j=2}^{2^{k+3}} \kappa_j (it / n^{1/2}) / j! \right) \]

(see (3.17)), the next to estimate

\[ \left( \frac{d}{dt} \right)^k \left[ \exp(n \{ \alpha(t / n^{1/2}) - 1 - \sum_{j=2}^{2^{k+3}} \mu_j (it / n^{1/2}) / j! \}ight. \]

\[ + \left. \sum_{j=2}^{2^{k+3}} \kappa_j (it / n^{1/2}) / j! \right) \]
\[ - \exp \left[ n \left( \alpha \left( t / n^{1/2} \right) - 1 - \sum_{j=1}^{2k+3} \mu_j (it / n^{1/2})^j / j! \right) + \sum_{j=1}^{2k+1} P_j (it)n^{-j/2} \right] e^{t^2/2} \]

(see (3.21)), and so on. The following lemma will prove useful. Define

\[ A_{nl}(t) = \alpha \left( t / n^{1/2} \right) - 1 - \sum_{j=1}^{2k+3} \mu_j (it / n^{1/2})^j / j! \]

**Lemma 3.1.** The following estimates are valid for all \( t > 0 \) and all integers \( 0 \leq \ell \leq 2k+2 \):

\[ |A_{nl}^{(\ell)}(t)| \leq C \left( 1 + t^2 \right) t^{2k+2-\ell} n^{(k+1)/2}, \quad (3.5) \]

\[ n |A_{nl}^{(\ell)}(t)| \leq C \left( 1 + t^2 \right) t^{2k+2-\ell} k \delta_n, \quad (3.6) \]

Furthermore, for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, whenever \( 0 < t \leq \delta n^{1/2} \),

\[ n |A_{nl}(t / n^{1/2})| \leq \epsilon t^2. \quad (3.7) \]

**Proof.** Since \( \delta_n = o(n^{-k}) \), it suffices to prove (3.6) and (3.7). In (3.6), we shall assume that \( \ell = 2m+1 \) is odd; the case of even \( \ell \) may be treated similarly. Now,

\[ |A_{nl}^{(\ell)}(t)| \leq |(d/dt)^{k} \mathbb{E} \{ \cos (tX / n^{1/2}) - \sum_{j=0}^{k+1} (-1)^j (tX / n^{1/2})^{2j} / (2j)! \} | \]

\[ + |(d/dt)^{k} \mathbb{E} \{ \sin (tX / n^{1/2}) \} \]

\[ - \sum_{j=0}^{k} (-1)^j (tX / n^{1/2})^{2j+1} / (2j+1)! \]

\[ - (d/dt)^{k} (-1)^{k+1} \mu_{2k+3} (t / n^{1/2})^{2k+3} / (2k+3)! | \]

\[ = |\mathbb{E} \{ (X / n^{1/2})^{k} \{ \sin (tX / n^{1/2}) \} \}

\[ - \sum_{j=0}^{k} (-1)^j (tX / n^{1/2})^{2j+1} / (2j+1)! \} | \]

\[ + |\mathbb{E} \{ (X / n^{1/2})^{k} \{ \cos (tX / n^{1/2}) \}

\[ - \sum_{j=0}^{k} (-1)^j (tX / n^{1/2})^{2j} / (2j)! \} | \]
\[ (-1)^{2k+3} \frac{t}{n^{\frac{1}{2}}} \frac{2k+3- \ell}{2(2k+3-\ell)} \frac{\mu_{2k+3}}{n^{\ell/2}} \]

\[ \leq 2\left[ E \{ \frac{X}{n^{\frac{1}{2}}} | tX/n^{\frac{1}{2}} ; \frac{1}{2} (x^{-m})^{2k+3} \} \right] \]

\[ + E \{ \frac{X}{n^{\frac{1}{2}}} | tX/n^{\frac{1}{2}} ; \frac{1}{2} (x^{-m})^{2k+3} \} \left( \frac{X}{n^{\frac{1}{2}}} \leq n^{\frac{1}{2}} \right) \]

\[ + n^{-(2k+3)/2} \left( \frac{X^{2k+3}}{n^{\frac{1}{2}}} \left( \frac{X}{n^{\frac{1}{2}}} \leq n^{\frac{1}{2}} \right) - \mu_{2k+3} \right) t^{2k+3-\ell} \]

\[ \leq 2 \left( 1 + t^2 \right) t^{2k+3-\ell} n^{\frac{1}{2}} \delta_n \]

as required for (3.6). To prove (3.7), observe that

\[ | \alpha(t) - 1 + t^2/2 | \leq E \{ \cos(tX) - 1 + (tX)^2/2 \} + E \{ \sin(tX) - tX \} \]

\[ = o(t^2) \]

as \( t \to 0 \). Therefore, given \( \epsilon > 0 \), we may choose \( \delta_n \in (0, 1) \) so small that

\[ n \left| \alpha \left( t/n^{\frac{1}{2}} \right) - 1 - \mu_2 \left( it/n^{\frac{1}{2}} \right)^2/2 \right| \leq \epsilon t^2/2 \]

for all \( 0 < t < \delta_n^{n^{\frac{1}{2}}} \). Result (3.7) follows on noting that if \( \delta \in (0, \delta_n^{n^{\frac{1}{2}}} \) is sufficiently small,

\[ n \left| \sum_{j=2}^{2k+3} \mu_j \left( it/n^{\frac{1}{2}} \right)^j/j! \right| \leq \delta t^2 \sum_{j=3}^{2k+3} \mu_j \left( it/n^{\frac{1}{2}} \right)^j/j! \leq \epsilon t^2/2 \]

whenever \( 0 < t < \delta n^{n^{\frac{1}{2}}} \).

Choose \( \epsilon > 0 \) so small that \( | \alpha(t) - 1 | \leq 1/2 \) for \( 0 < t \leq 2 \epsilon \). Define

\[ A_{n2}(t) = n \log \alpha \left( t/n^{\frac{1}{2}} \right) = n \sum_{j=1}^{\infty} (-1)^{j+1} \left( \alpha \left( t/n^{\frac{1}{2}} \right) - 1 \right)^j/j. \]

Then for \( 0 < t < \epsilon n^{n^{\frac{1}{2}}} \),

\[ A_{n3}(t) = \left| (d/dt)^j [ A_{n2}(t) - n \sum_{j=1}^{3k-4} (-1)^{j+1} \left( \alpha \left( t/n^{\frac{1}{2}} \right) - 1 \right)^j/j ] \right| \]

\[ \leq n \sum_{j=1}^{\infty} \frac{1}{j^{j-1}} \left| (d/dt)^j \left( \alpha \left( t/n^{\frac{1}{2}} \right) - 1 \right)^j \right|. \]
Since $\alpha$ has $2k + 2$ bounded derivatives, and $|\alpha(t) - 1| \leq t^2 / 2$, then for $j \geq 3k + 4$ and $0 \leq \ell \leq 2k + 2$,

$$
| (d/dt)^{\ell} \{\alpha(t) - 1\} / | \leq C_j^{\ell} |\alpha(t) - 1|^{j - \ell} \\
\leq C_j^{\ell} (t^2 / 2)^{k + 2} |\alpha(t) - 1|^{j - \ell - (k + 2)},
$$

where $C$ does not depend on $j$. Therefore if $0 < t < \varepsilon n^{1/2}$,

$$
A_{n3}(t) \leq C_i n n^{-\ell/2} (t^2 / 2n)^{k + 2} \sum_{j=0}^{\infty} (j + 3k + 4)^{\ell} (1/2)^j \\
\leq C_2 t^{2k + 2} n^{\ell - (k + 1)}
$$

(3.8)

Next, observe that

$$
n^{-1} A_{n4}(t) \equiv | (d/dt)^{\ell} \left[ \sum_{r=2}^{3k + 3} (-1)^{r+1} \{\alpha(t/n^{1/2}) - 1\} \right] / r \\
- \sum_{r=2}^{3k + 3} (-1)^{r+1} \left\{ \sum_{j=1}^{2k + 3} \mu_j (it/n^{1/2}) / j! \right\} r / r | \\
= | (d/dt)^{\ell} \left[ \sum_{r=2}^{3k + 3} (-1)^{r+1} r^{-1} \sum_{s=1}^{r} \{A_{n1}(t)\}^{s} \\
\times \left\{ \sum_{j=2}^{2k + 3} \mu_j (it/n^{1/2}) / j! \right\} r^{-s} \right| \\
\leq C_i \sum_{r=2}^{3k + 3} \sum_{s=1}^{r} \left| (d/dt)^{\ell-2} \left[ \sum_{j=1}^{2k + 3} \mu_j (it/n^{1/2}) / j! \right] r^{-s} \right| \\
\leq C_3 (1 + t^9) t^{2k + 2 - \ell} n^{\ell - (k + 2)},
$$

(3.9)

using (3.5). By definition of the cumulants $\kappa_j$,

$$
\sum_{r=1}^{3k + 3} (-1)^{r+1} \left\{ \sum_{j=1}^{2k + 3} \mu_j (it)^j / j! \right\} r / r = \sum_{j=2}^{2k + 3} \kappa_j (it)^j / j! + A(t),
$$

where $A(t)$ denotes a sum of constant multiples of $(it)^{2k + 4}$, $(it)^{2k + 5}$, \ldots, $(it)^{(2k + 3)(3k + 3)}$. Thus,

$$
A_{n5}(t) = n | (d/dt)^{\ell} \left[ \sum_{r=1}^{3k + 3} (-1)^{r+1} \left\{ \sum_{j=1}^{2k + 3} \mu_j (it/n^{1/2}) / j! \right\} \right] / r \\
= n | (d/dt)^{\ell} \left[ \sum_{r=1}^{3k + 3} (-1)^{r+1} \left\{ \sum_{j=1}^{2k + 3} \mu_j (it/n^{1/2}) / j! \right\} \right] / r \\
\leq C_4 (1 + t^9) t^{2k + 2 - \ell} n^{\ell - (k + 2)},
$$
\[ -\sum_{r=2}^{2k+3} \kappa_j (it / n^{1/2}) / j! \mid \]
\[ \leq C_1 n^{- (k+1)} \sum_{r=2k+4}^{(2k+3)(3k+3)} t^{-r} \]
\[ \leq C_2 (1 + t^p) t^{2(k+2)-\ell} n^{-(k+1)}. \tag{3.10} \]

for all \( t > 0 \). Combining (3.8), (3.9) and (3.10), we see that

if \( 0 < t < \varepsilon n^{1/2} \) and \( 0 \leq \ell \leq 2k+2 \),

\[ \mid (d/dt)^{\ell} \{ n \log \alpha (t / n^{1/2}) - n \{ \alpha (t / n^{1/2}) - 1 - \sum_{j=2}^{2k+3} \mu_j (it / n^{1/2}) / j! \}
\[ - n \sum_{j=2}^{2k+3} \kappa_j (it / n^{1/2}) / j! \mid \]
\[ \leq A_{n3} (t) + A_{n4} (t) + A_{n5} (t) \]
\[ \leq C (1 + t^p) t^{2(k+2)-\ell - (k+1)} n^{-(k+1)}. \tag{3.11} \]

Let

\[ A_{n6} (t) = n \{ \alpha (t / n^{1/2}) - 1 - \sum_{j=2}^{2k+3} \mu_j (it / n^{1/2}) / j! + \sum_{j=2}^{2k+3} \kappa_j (it / n^{1/2}) / j! \} \]

and

\[ A_{n7} (t) = n \log \alpha (t / n^{1/2}) - A_{n6} (t). \]

Then result (3.11) may be written as

\[ \mid (d/dt)^{\ell} A_{n7} (t) \mid \leq C (1 + t^p) t^{2(k+2)-\ell} n^{-(k+1)}. \tag{3.12} \]

we may deduce from (3.9) that

\[ \mid (d/dt)^{\ell} A_{n6} (t) \mid \leq C (1 + t^p). \tag{3.13} \]

Let \( \Sigma_{(\ell, i)} \) denote summation over vectors \((i_1, \ldots, i_r)\)

with \( 1 \leq r \leq \ell \), \( i_s \geq 1 \) for each \( a \), and \( \sum_{s=1}^{r} i_a = \ell \). Let \( \Sigma_{(\ell, x)} \)

denote summation over vectors \((i_1, \ldots, i_r)\) and \((i_1, \ldots, i_s)\)

with \( 0 \leq r \leq \ell - 1 \), \( 1 \leq s \leq \ell \), \( i_s \geq 1 \) and \( j_s \geq 1 \) for each \( a \),
\[
\sum_{s=1}^{r} i_s + \sum_{s=1}^{s} j_s = \ell. \text{ Combining (3.12) and (3.13), we see that}
\]
\[
A_{n^6}(t) = |(d/dt)^{\ell}\left[\alpha^n(t/n^{1/2}) - \exp\{A_{n^6}(t)\}\right]|
\]
\[
= |(d/dt)^{\ell}\exp\{A_{n^6}(t)\} \cdot 1 - \exp\{A_{n^7}(t)\}| \leq C_1 \left(\sum_{s=1}^{s} A_{n^6}^{(s)}(t) \cdots A_{n^6}^{(s)}(t) \cdots \cdot \cdot \cdot \exp\{A_{n^6}(t)\} \right)
\]
\[
\cdot \left[1 - \exp\{A_{n^7}(t)\}\right] \leq C_2 (1 + t^2) \left(\sum_{s=1}^{s} \exp\{A_{n^6}(t)\} \cdot 1 - \exp\{A_{n^7}(t)\}\right)
\]
\[
+ t^{2k+2-\ell} \sum_{s=1}^{s} \exp\{A_{n^6}(t)\} \cdot 1 - \exp\{-A_{n^7}(t)\}\right)
\]
\[
= C_2 (1 + t^2) |\alpha^n(t/n^{1/2})| \left[1 - \exp\{-A_{n^7}(t)\}\right]
\]
\[
+ t^{2k+2-\ell} \cdot n^{-(k+1)} \cdot n^{-(k+1)} \right). \tag{3.14}
\]

In view of (3.12) (with \(\ell = 0\)), there exists \(\delta \in (0, 1/2)\) such that
\[
|A_{n^6}(t)| \leq C t
\]
whenever \(0 < t \leq n^\delta\) and \(n\) is sufficiently large. For such values of \(t\),
\[
|1 - \exp\{-A_{n^7}(t)\}| < \cdot A_{n^7}(t) \exp\{|A_{n^7}(t)|\}
\]
\[
\leq C_1 t^{2(k+2)-\ell} \exp(C_2 t) n^{-(k+1)} \cdot \tag{3.15}
\]

Furthermore,
\[
|\alpha(t)|^2 = \left[1 - E\{1 - \cos(tx)\}\right]^2 + E\{\sin(tx) - tx\}^2.
\]

As \(t \to 0\), \(E\{1 - \cos(tx)\} \sim t^2/2\) and \(E(\sin(tx) - tx) = o(t^2)\).
Hence there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$|a(t)| \leq 1 - 2\varepsilon_1 t^2$$

whenever $0 < t \leq \varepsilon_2$. Consequently,

$$|a^n(t/n^{\frac{1}{2}})| \leq \exp(-2\varepsilon_1 t^2)$$  \hspace{1cm} \text{(3.16)}

if $0 < t \leq \varepsilon_2 n^{\frac{1}{8}}$. Combining (3.14), (3.15) and (3.16) we see that if $0 < t < n^\delta$ and $n$ is sufficiently large,

$$A_{n8}(t) \leq C n^{-\frac{2}{5}} \exp(-\varepsilon_1 t^2) n^{-(k+1)}.$$  \hspace{1cm} \text{(3.17)}

Next, observe that

$$A_{n9}(t) \equiv |(d/dt)^{\ell} [\exp \{A_{n6}(t)\} - \exp \{nA_{n1}(t)\}]$$

$$\cdot \left[ 1 + \frac{2k+1}{n^{\frac{1}{2}}} e^{-i^2/2} \right] |$$

$$\leq \sum_{\alpha=6}^{\ell} \binom{\ell}{\alpha} |(d/dt)^{\ell-\alpha} \left[ \exp \{n \sum_{j=3}^{2k+3} \kappa_j (it/n^{\frac{1}{2}}) \} \right]|$$

$$\times |(d/dt)^{\ell-\alpha} \left[ \exp \left\{ n \sum_{j=1}^{2k+1} P_j(it/n^{\frac{1}{2}}) \right\} e^{-i^2/2} \right] (3.18)$$

Now,

$$\exp \{n \sum_{j=3}^{2k+3} \kappa_j (it/n^{\frac{1}{2}}) \} = 1 + \sum_{j=1}^{2k+1} P_j(it) n^{-j/2}$$

$$+ \sum_{r=2k+2}^{\infty} \{ n \sum_{j=3}^{2k+3} \kappa_j (it/n^{\frac{1}{2}}) \}^r/r$$

$$+ n (it/n^{\frac{1}{2}})^{2(k+2)} A_{n10}(t),$$

where $A_{n10}(t)$ denotes a polynomial in it of degree not exceeding $(2k+1)(2k+3) - 2(k+2)$ and all of whose coefficients are uniformly bounded. Therefore
\[ A_{n,11}(t) = |(d/dt)^f \left[ \exp \left\{ n \sum_{j=3}^{2k+3} (it/n^{1/2})^j/j! \right\} \right. \]
\[ - \left\{ 1 + \sum_{j=1}^{2k+1} P_j(it) n^{-j/2} \right\} | \]
\[ \leq \sum_{r=2k+2}^{\infty} \frac{(d/dt)^f \left\{ n \sum_{j=3}^{2k+3} (it/n^{1/2})^j/j! \right\} \right\}^r / r! \]
\[ + C (1 + t^p) t^{2(2k+2)} n^{-(k+1)} \]
\[ \sum_{r=1}^{\infty} \left| \left( \frac{d}{dt} \right)^{r} \{ nA_{n_1} (t) \} \right|^r \]

\[ \cdot \left\{ 1 + \sum_{j=1}^{2k+1} p_j (it) n^{-j/2} e^{-t^2/2} \right\} / r! \]

\[ \leq C_1 (1 + t^p) e^{-t^2/2} \sum_{r=2}^{\infty} (r!)^{-1} \sum_{a=0}^{\ell} \left| \left( \frac{d}{dt} \right)^a \{ nA_{n_1} (t) \} \right|^r . \]  \( \text{(3.22)} \)

Now,

\[ A_{n_{13}} (t) \equiv \sum_{a=0}^{\ell} \left| \left( \frac{d}{dt} \right)^a \{ nA_{n_1} (t) \} \right|^r \]

\[ \leq C r^\ell \left\{ \left| nA_{n_1} (t) \right|^{r-2} + \left| nA_{n_1} (t) \right|^{\max \{ 0, r-2-\ell \}} \right\} \times \]

\[ \times \left\{ \left| nA_{n_1} (t) \right|^{2} + \left| nA_{n_1} (t) \right|^{2} \sum_{a=1}^{\ell} \left| nA_{n_1} (t) \right|^{a} \right\} \]

\[ + \sum_{a=2}^{\ell} \sum_{(a_3)} \left| nA_{n_1} (t) \right|^{a_3} \ldots \ldots \left| nA_{n_1} (t) \right|^{a_s} \]

where \( C \) does not depend on \( r \), and \( \sum_{(a_3)} \) denotes summation over vectors \((j_1, \ldots, j_s)\) with \( 2 \leq s \leq a \), each \( j_b \geq 1 \), and \( \sum_{b=1}^{s} j_b = a \). Using (3.6) and (3.7) we see that if \( 0 < t < \varepsilon n^{1/2} \) and \( \varepsilon \) is sufficiently small,

\[ A_{n_{13}} (t) \leq C r^\ell \left\{ \left( t^2 / 10 \right)^{r-2} + \left( t^2 / 10 \right)^{\max \{ 0, r-2-\ell \}} \right\} \left\{ (1 + t^p) t^{2k+3-\ell} \delta_n \right\} . \]

where \( C \) does not depend on \( r \). Substituting into (3.22), we obtain

\[ A_{n_{12}} (t) \leq C t^{2k+3-\ell} \exp (-t^2/4) \delta_n^2 . \]  \( \text{(3.23)} \)

In view of (3.6),

\[ A_{n_{14}} (t) = \left| \left( \frac{d}{dt} \right)^{r} \left\{ 1 + nA_{n_1} (t) \right\} \left\{ 1 + \sum_{i=1}^{2k+1} p_i (it) n^{-i/2} e^{-t^2/2} \right\} \right| \]

\[ - \left\{ 1 + \sum_{i=1}^{2k+1} p_i (it) n^{-i/2} + nA_{n_1} (t) e^{-t^2/2} \right\} \right| \]

\[ \leq C (1 + t^p) t^{2k+3-\ell} \exp (-t^2/2) n^{1/2} \delta_n , \]  \( \text{(3.24)} \)

uniformly in \( t \). (Note that the polynomials \( p_i (it) \) satisfy \( p_i (0) = 0 \).) Combining (3.17), (3.21), (3.23) and (3.24), we see
that if \( 0 < t < n^{\delta} \) and \( n \) is sufficiently large,
\[
| a_n^{(i)}(t) | = | (d/dt)^{i} [a_n^{(i)}(t/n^{\delta/2}) - \{ 1 + \sum_{j=1}^{2k+1} P_j(it) n^{-j/2} 
+ nA_{n,k}(t) e^{-i^2/2} \} | 
\leq A_{n,k}(t) + A_{n,9}(t) + A_{n,12}(t) + A_{n,14}(t) 
\leq C t^{2k+3-\ell} \exp (-\varepsilon t^2) (n^{-k+1} + \delta_n^{2} + n^{-\eta} \delta_n^{2}),
\]
(3.25)

To treat the case where \( t > n^{\delta} \), observe that
\[
| a_n^{(i)}(t) | \leq | (d/dt)^{i} a_n^{(i)}(t/n^{\delta/2}) | + | (d/dt)^{i} \{ 1
+ \sum_{j=1}^{2k+1} P_j(it) n^{-j/2} + nA_{n,k}(t) e^{-i^2/2} \} | 
\leq C (1 + t^p) n^p \{ | a_n^{(i)}(t/n^{\delta/2}) |^{n-\ell} + e^{-i^2/2} \},
\]
(3.26)
for all \( t > 0 \). It follows from (3.16) that for some
\( \varepsilon \in (0, 1/4) \) and \( \eta > 0 \),
\[
| a_n^{(i)}(t/n^{\delta/2}) |^{n-\ell} \leq C \exp (-2\varepsilon t^2)
\]
whenever \( 0 < t < \eta n^{\delta/2} \). Consequently,
\[
| a_n^{(i)}(t) | \leq C t^{2k+3-\ell} \exp (-\varepsilon t^2) n^{-(k+1)}
\]
uniformly in \( n^{\delta} < t < n^{\delta/2} \). Therefore (3.25) holds uniformly in
\( 0 < t < \eta n^{\delta/2} \).

Let
\[
\rho = \sup_{t > \eta} | a(t) | < 1.
\]
If \( \eta n^{\delta/2} < t < n^{k+1} \), it follows from (3.26) that
\[
| a_n^{(i)}(t) | \leq C_2 (1 + t^p) n^p (\varphi_n + e^{-i^2/2}) < C_3 \varphi_n
\]
(3.27)
for some \( r < 1 \). (Here \( C \) does not depend on \( t \).)

Returning to (3.1), taking \( T = n^{k+1} \), and estimating
\[
\left| \sum_{k} d_{n}^{(k)}(t) \right| \quad \text{using (3.25) if } 0 < t < \eta n^{\frac{1}{2}} \text{, or using (3.27) if } \eta n^{\frac{1}{2}} < t < n^{k+1},
\]
we see that
\[
\sup_{x \to \pm \infty} \left( 1 + |x|^{2k+2} \right) |D_{n}(t)| = O\left(n^{-(k+1)} + \delta_{n}^{2} + n^{-\frac{1}{2}} \delta_{n}^{2}\right),
\]
which proves Theorem 1.

**PROOF OF THEOREM 2.** Define
\[
\Delta = \Delta(x, X/n^{\frac{1}{2}}) = \Phi(x - X/n^{\frac{1}{2}}) - \Phi(x) - \sum_{j=1}^{2k+2} (-X/n^{\frac{1}{2}})^{j} \frac{\phi^{(j-1)}(x)}{j!}
\]
\[
= \left(-X/n^{\frac{1}{2}}\right)^{2k+3} \phi^{(2k+2)}(x)/(2k+3)!
\]
\[
+ \left(-X/n^{\frac{1}{2}}\right)^{2k+4} \phi^{(2k+3)}(x - \theta X/n^{\frac{1}{2}})/(2k+4)!,
\]
where \( 0 \leq \theta = \theta(x, X/n^{\frac{1}{2}}) \leq 1 \). If \( x \geq 0 \) then
\[
\left| E\{|\Delta I(|X/n^{\frac{1}{2}}| \leq 1)|\} + \mu_{2k+3} \eta^{-\frac{2k+3}{2}} \phi^{(2k+2)}(x)/(2k+3)! \right|
\leq \left| \phi^{(2k+2)}(x) \right| \left| E\{ (X/n^{\frac{1}{2}})^{2k+3} I(|X/n^{\frac{1}{2}}| \leq 1) \} - \mu_{2k+3} n^{-\frac{2k+3}{2}} \right|
\]
\[
+ \sup_{x \to \pm \infty, y \to \pm 1} \left| \phi^{(2k+3)}(y) \right| \left| E\{ (X/n^{\frac{1}{2}})^{2k+4} I(|X/n^{\frac{1}{2}}| \leq 1) \} \right|,
\]
(3.28)

if \( 0 \leq x \leq 2 \),
\[
\left| E\{\Delta I(|X/n^{\frac{1}{2}}| > 1)\} \right|
\leq P\{ |X/n^{\frac{1}{2}}| > 1 \} + \sum_{j=1}^{2k+2} \left| \phi^{(j-1)}(x) \right| \left| E\{ |X/n^{\frac{1}{2}}| I(|X/n^{\frac{1}{2}}| > 1) \} \right|
\leq \left(1 + \sum_{j=1}^{2k+2} \left| \phi^{(j-1)}(x) \right| \right) \left| E\{ |X/n^{\frac{1}{2}}|^{2k+2} I(|X/n^{\frac{1}{2}}| > 1) \} \right|,
\]
and if $x > 2$,

$$| E \{ \Delta \mid 1 < \mid X/n^{1/3} \mid \leq x/2 \} |$$

$$\leq \{ \Phi (x) - \Phi (x/2) \} P (\mid X \mid > n^{1/3})$$

$$+ \sum_{j=1}^{2k+2} | \phi^{(j-1)} (x) | E \{ \mid X/n^{1/3} \mid ^j I (\mid X/n^{1/3} \mid > 1) \}$$

$$\leq \{ 1 - \Phi (x/2) \}$$

$$+ \sum_{j=1}^{2k+2} | \phi^{(j-1)} (x) | E \{ \mid X/n^{1/3} \mid ^{2k+2} I (\mid X/n^{1/3} \mid > 1) \}$$

(3.30)

and

$$| E \{ \Delta \mid \mid X/n^{1/3} \mid > x/2 \} |$$

$$\leq P (\mid X/n^{1/3} \mid > x/2)$$

$$+ \sum_{j=1}^{2k+2} | \phi^{(j-1)} (x) | E \{ \mid X/n^{1/3} \mid ^{2k+2} I (\mid X/n^{1/3} \mid > 1) \}$$

$$\leq \{ (2/x)^{2k+2} \}$$

$$+ \sum_{j=1}^{2k+2} | \phi^{(j-1)} (x) | E \{ \mid X/n^{1/3} \mid ^{2k+2} I (\mid X/n^{1/3} \mid > 1) \} \}$$

(3.31)

From (3.28) and (3.29) we see that for $0 \leq x \leq 2$,

$$\Delta^* = \Delta^* (x)$$

$$= n \mid E (\Delta) \mid + \mu_{2k+3} n^{-(2k+3)/2} \phi^{(2k+3)} (x)/(2k+3)! |$$

$$\leq C \kappa \delta_n,$$

while from (3.28), (3.30) and (3.31) we obtain for $x > 2$,

$$\Delta^* \leq \{ 1 - \Phi (x/2) \} + (2/x)^{2k+2} + \sum_{j=0}^{2k+2} \mid \phi^{(j)} (x) \mid$$

$$+ \sup_{x/1 \leq y \leq x+1} \mid \phi^{(2k+3)} (y) \mid \times \kappa \delta_n.$$

(3.32)
Therefore

\[ \sup_{x \geq 0} (1 + x^{2k+2}) \Delta^* (x) \leq C_k \delta_n, \]

and a similar result may be proved for negative \( x \).

**PROOF OF THEOREM 3.** It suffices to establish (2.5) in the case \( p = 1 \). Suppose there exists a sequence \( n_j \to \infty \) such that

\[ k \delta_n^{-1} \int_0^\xi |L_{n_j}(x)| \, dx \to 0 \quad (3.33) \]

as \( j \to \infty \). We shall deduce a contradiction.

Define

\[
k \delta_n (\xi) \equiv n^{-k} E \{ X^{2k+2} I (|X| > n^{\frac{1}{2}} \xi) \}
+ n^{-k} E \{ X^{2k+4} I (|X| \leq n^{\frac{1}{2}} \xi) \}
+ n^{-k} E \{ X^{2k+3} I (|X| \leq n^{\frac{1}{2}} \xi) \} - \mu_{2k+3},
\]

for \( \xi > 0 \). If \( 0 < \xi \leq 1 \) then \( k \delta_n \leq 3 k \delta_n (\xi) \), and so (3.33) is equivalent to

\[ \{ k \delta_n (\xi) \}^{-1} \int_0^\xi |L_{n_j}(x)| \, dx \to 0 \quad (3.34) \]

whenever \( 0 < \xi \leq 1 \). We consider two cases separately.

(i) Suppose that for each \( 0 < \xi_o \leq 1 \) there exists

\( \xi \in (0, \xi_o] \) such that

\[
\begin{align*}
\left[ n_j^{-k} E \{ X^{2k+4} I (|X| \leq \xi n_j^{\frac{1}{2}}) \} \\
+ n_j^{-k} E \{ X^{2k+3} I (|X| \leq \xi n_j^{\frac{1}{2}}) \} - \mu_{2k+3} \right] \\
\times \left[ n_j^{-k} E \{ X^{2k+2} I (|X| > \xi n_j^{\frac{1}{2}}) \} \right]^{-1} \to \infty
\end{align*}
\]

as \( j \to \infty \) along a suitable subsequence (which may depend on \( \xi \)).
Choose $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon)$ such that

$$\{ \int_0^{\xi_{(2k+2)}} \phi^{(2k+2)}(x) / (2k+2)! \, dx \} \times \{ \int_0^{\xi_{(2k+3)}} \phi^{(2k+3)}(x) / (2k+3)! \, dx \}$$

$$- \{ \int_0^{\xi_{(2k+2)}} \phi^{(2k+2)}(x) / (2k+2)! \, dx \} \times \{ \int_0^{\xi_{(2k+3)}} \phi^{(2k+3)}(x) / (2k+3)! \, dx \} \equiv 2a > 0,$$

and then select $\xi_o \in (0, 1)$ such that

$$\{ \int_0^{\xi_{(2k+2)}} \phi^{(2k+2)}(x) / (2k+2)! \, dx \}$$

$$\times \int_0^{\xi_{(2k+3)}} \inf_{|x-y| < \xi_o} \phi^{(2k+3)}(y) / (2k+3)! \, dy \}$$

$$- \{ \int_0^{\xi_{(2k+2)}} \phi^{(2k+2)}(x) / (2k+2)! \, dx \} \times \int_0^{\xi_{(2k+3)}} \sup_{|x-y| < \xi_o} \phi^{(2k+3)}(y) / (2k+3)! \, dy \} \geq a. \quad (3.36)$$

Choose $\xi \leq \min(\varepsilon, \xi_o)$ such that (3.35) holds along a subsequence $\{m_j\} \subseteq \{n_j\}$.

Then it may be proved that for each $\eta \in (0, \varepsilon)$,

$$b_j(\eta) = \int_0^\eta \left| k \cdot L_{m_j} (x) \right| dx$$

$$\geq m_j^{-(2k+1)/2} \left| E \left\{ X_{2k+3}^\eta \, I \left( |X| \leq \xi m_j^\eta \right) \right\} - \mu_{2k+3} \right| I_1(\eta)$$

$$- m_j^{-k+1} \left| E \left\{ X_{2k+4}^\eta \, I \left( |X| \leq \xi m_j^\eta \right) \right\} \right| I_2(\eta, j) \right|$$

$$+ o (k \delta_{m_j}). \quad (3.37)$$

where

$$I_1(\eta) = \int_0^\eta \phi^{(2k-2)}(x) / (2k+3)! \, dx,$$

$$I_2(\eta, j) = \int_0^\eta \phi^{(2k-3)}(x) / (2k+4)! \, dx.$$
and $x'$ denotes a function of $x$ and $j$ taking values between $x - \xi$ and $x + \xi$. Consequently,

$$I_2(\varepsilon_j, j) b_j(\varepsilon_j) + I_2(\varepsilon_1, j) b_j(\varepsilon_1)$$

$$\geq m_j^{-(2k+1)/2} | \mathbb{E}\{X^{2k+3} I(|X| \leq \xi m_j^{1/2})\} - \mu_{2k+3} |$$

$$\cdot | I_2(\varepsilon_j, j) I_1(\varepsilon_1) - I_2(\varepsilon_1, j) I_1(\varepsilon_2) + o(\delta_{m_j})$$

$$\geq m_j^{-(2k+1)/2} | \mathbb{E}\{X^{2k+3} I(|X| \leq \xi m_j^{1/2})\} - \mu_{2k+3} | \cdot a + o(\delta_{m_j}),$$

the second inequality following (3.36). Likewise,

$$I_1(\varepsilon_j) b_j(\varepsilon_j) + I_1(\varepsilon_1) b_j(\varepsilon_1)$$

$$\geq m_j^{-(k+1)} \mathbb{E}\{X^{2k+4} I(|X| \leq \xi m_j^{1/2})\} \cdot a + o(\delta_{m_j}).$$

These inequalities, together with (3.35), provide a contradiction to (3.34).

(ii) Suppose the conditions of part (i) do not obtain. Then for all sufficiently small positive $\xi$,

$$\limsup_{j \to \infty} \frac{n_j^{-(2k+1)/2} | \mathbb{E}\{X^{2k+3} I(|X| \leq \xi n_j^{1/2})\} - \mu_{2k+3} |}{[n_j^{-k} \mathbb{E}\{X^{2k+2} I(|X| > \xi n_j^{1/2})\}]} < \infty. \quad (3.38)$$

Now, $\Phi^{(2k+3)}(x) = (-1)^k c_k x + o(|x|^3)$ as $x \to 0$, where $c_k > 0$. It may be proved after some lengthy algebra that for this definition of $c_k$,

$$L_n(2x) - L_n(x)$$

$$= nx \mathbb{E}\{[\Phi(X/n^{1/2}) - \sum_{\ell=0}^{k-1} (X/n^{1/2})^{2\ell} \Phi^{(2\ell)}(0) / (2\ell)!]$$

$$\cdot I(|X| > \xi n^{1/2})\}$$
\[ + n^{-(k+1)} E \{ X^{2k+4} I \left( \mid X \mid \leq n^{\frac{k}{2}} \right) \} \left( -1 \right)^k c_k x / (2k+4)! \] 
\[ + n \left[ (2k+1)^{\frac{1}{2}} \mu_{2k-3} - E \{ X^{2k+3} I \left( \mid X \mid \leq \xi n^{\frac{k}{2}} \right) \} \right] \] 
\[ \cdot \left[ \Phi^{(2k+3)}(x) - \Phi^{(2k+2)}(x) \right] / (2k+3)! \] 
\[ + A_n, \]  
(3.39)

where for a constant \( C_1 \) depending only on \( k \),
\[ |A_n| \leq C_1 \left( |x|^3 + \xi |x| + \xi^{\frac{k+1}{2}} n^{(k+1)} E \{ X^{2k+4} I \left( \mid X \mid \leq \xi n^{\frac{k}{2}} \right) \} \right) \]
\[ + C_1 \left( x^2 \xi^{-2k} + |x|^3 \xi^{-2(k+1)} n^{-k} E \{ X^{2k+2} I \left( \mid X \mid > \xi n^{\frac{k}{2}} \right) \} \right) \]

uniformly in \( |x| \leq 1 \) and \( \xi > 0 \). Now, the continuous function
\[ g(u) = (-1)^k \left\{ \Phi(u) - \sum_{\ell=0}^{k+1} u^{2\ell} \Phi^{(2\ell)}(0) / (2\ell)! \right\} \]
\[ = (-1)^k (2\pi)^{-\frac{k}{2}} \left\{ e^{-u^2/2} - \sum_{\ell=0}^{k+1} (-u^2/2)^\ell / \ell! \right\}, \]
is positive on \( (-\infty, \infty) - \{ 0 \} \) and satisfies
\[ g(u) \geq d_k u^{2k+2} \min(1, u^2) \geq d_k \xi^2 u^{2k+2} \]  
(3.40)

for all \( |u| > \xi \) and \( \xi \in (0, 1) \), where \( d_k \) is a positive constant. Therefore we may integrate the expansion (3.39) to obtain the following result: for a positive constant \( C_2 \) and all \( \xi, \eta \in (0, 1) \),
\[ 2 \int_0^\eta \left| \frac{L_n}{k_n} \right| (2x) - \frac{L_n}{k_n} (x) \mid dx \]
\[ \geq n\eta^2 E \left\{ g \left( X / n^{\frac{k}{2}} \right) I \left( \mid X \mid > \xi n^{\frac{k}{2}} \right) \} \]
\[ + n^{-(k+1)} E \{ X^{2k+4} I \left( \mid X \mid \leq n^{\frac{k}{2}} \right) \} c_k \eta^2 / (2k+4)! \]
\[
\begin{align*}
- C_2 \left( \eta^4 + \xi \eta^2 + \xi^4 \right) n^{- (k+1)} E \left\{ X^{2k-4} I \left( |X| \leq \xi n^{1/2} \right) \right\} \\
- C_2 \left( \xi^{-2k} \eta^3 + \xi^{-2} \eta^4 \right) n^{-k} E \left\{ X^{2k+2} I \left( |X| > \xi n^{1/2} \right) \right\} \\
- C_2 n^{-(2k+1)/2} \cdot E \left\{ X^{2k+3} I \left( |X| \leq \xi n^{1/2} \right) \right\} - \mu_{2k+3} | \eta^3 \\
\geq n^{-k} E \left\{ X^{2k+2} I \left( |X| > \xi n^{1/2} \right) \right\} \left\{ d_k \xi^2 \eta^2 \right\} \\
- C_2 \left( \xi^{-2k} \eta^3 + \xi^{-2} \eta^4 \right) \\
+ n^{- (k+1)} E \left\{ X^{2k+4} I \left( |X| \leq \xi n^{1/2} \right) \right\} \\
\cdot \left\{ c_k \eta^2 / (2k+4)! - C_2 \left( \eta^4 + \xi \eta^2 + \xi^4 \right) \right\} \\
- C_2 \eta^3 n^{-(2k+1)/2} \cdot E \left\{ X^{2k+3} I \left( |X| \leq \xi n^{1/2} \right) \right\} - \mu_{2k+3} | \eta^3 ,
\end{align*}
\]

the last inequality following from (3.40). Taking $\xi = c \eta^{1/2} (k+1)$ for a positive constant $c$, and replacing $n$ by $n_j$, we find that for small values of $\eta$,

\[
2 \int_0^n \left| k \cdot L_{n_j} (2x) - \kappa \cdot L_{n_j} (x) \right| \, dx \\
\geq n_j^{-k} E \left\{ X^{2k+2} I \left( |X| > \xi n_j^{1/2} \right) \right\} \left\{ d_k c^2 \eta^2 + \eta^3 \right\} - C_2 \left\{ c \cdot 2k + 2^1 (k+1) \right\} \\
+ c^{-2} \eta^3 + \rho_n \left( \xi \eta^3 \right) \right] + n_j^{- (k+1)} E \left\{ X^{2k+4} I \left( |X| \leq \xi n_j^{1/2} \right) \right\} \\
\times \eta^2 \left\{ c_k / (2k+4)! - C_2 \left( \eta^2 + c \eta^{1/2} (k+1) + c^4 (k+1) \eta \right) \right\} ,
\]

(3.41)

where

\[
\rho_n (x) = n^{- (2k+1)/2} \left\{ E \left\{ X^{2k+3} I \left( |X| \leq \xi n_j^{1/2} \right) \right\} - \mu_{2k+3} \right\} / \left\{ n^{-k} E \left\{ X^{2k+2} I \left( |X| > \xi n_j^{1/2} \right) \right\} \right\} .
\]

A little algebra shows that if $0 < \xi \leq \epsilon$ then

\[
\rho_n (x) \leq \rho_n (\epsilon) + \epsilon ,
\]

and it follows from (3.38) that if $\epsilon < 1$ is sufficiently small,
\[ s \equiv \limsup_{n \to \infty} \rho_n(\varepsilon) < \infty. \]

In this case we may be sure that if \( c \eta^{1/2(k+1)} < \varepsilon \), and \( j \) is large, the term within square brackets in (3.41) is not less than

\[ d_k c^2 \eta^{2+1/(k+1)} - C_2 \{ c^{-2k} \eta^{2+1/(k+1)} + c^{-2} \eta^{1/(k+1)}(s+\varepsilon+1) \eta^3 \}. \]

Choose \( c > 1 \) so large that

\[ d_k c^2 - C_2 \{ c^{-2k} + c^{-2} \eta^{1/(k+1)} + (s+\varepsilon+1) \} \geq 1, \]

and let \( \eta \) be so small that \( \eta \leq 1 \), \( c\eta^{1/2(k+1)} \leq \varepsilon \) and

\[ c_k / (2k+4)! - C_2 (\eta^2 + c\eta^{1/2(k+1)} + c^{4/(k+1)} \eta) > c_k / 2 \cdot (2k+4)!. \]

Then by (3.41), for large \( j \),

\[
2 \int_{-\infty}^{\eta_j} | L_{n_j} (2x) - L_{n_j} (x) | \, dx \\
\geq n_j^{-k} E \{ X^{2k+2} I (|X| > \xi n_j^{1/2}) \} \eta^{2+1/(k+1)} \\
+ n_j^{-(k+1)} E \{ X^{2k+4} I (|X| \leq \xi n_j^{1/2}) \} \eta^2 c_k / 2 \cdot (2k+4)!. \]

This estimate and (3.38) lead to a contradiction of (3.34), and so complete the proof of Theorem 3.

**REFERENCES**


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