Local Finiteness of Analytic Arcs

Yukio Hirashita
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In this paper we shall define the number of the analytic arcs of the Weierstrass points on open Riemann surface.

1. Let \( f \) be a function of the unit open disk \( \{ z = x + iy \mid |z| < 1 \} \) such that \( f(z) = f(x, y) \) is a non-zero analytic function with respect to two real variables \( x \) and \( y \). It is a well-known result that \( \{ z \mid f(z) = 0 \} \) is composed of at most countable number of branches and its complement is open dense [4, p. 439; 7, p. 252; 10, p. 508]. We shall prove the following sharper phenomenon.

THEOREM 1. For any point \( p \) of the unit open disk there exists a neighborhood \( U \) of \( P \) such that \( U \cap \{ z \mid f(z) = 0 \} \) is a finite sum of analytic arcs.

It should be noted that, in this paper, an arc is called an analytic arc if it is the image of an analytic function of an open interval on the real axis. For example, a point is the image of a constant analytic function. In our proof some results on 1-dimensional irreducible analytic sets will play a central role.

A meromorphic function \( f \) on an open Riemann surface \( R \) of finite genus \( g \) is called a rational function if the differential \( \text{Re } df \) is distinguished. A point \( p \in R \) is called a Weierstrass point if there exists a non-constant rational function whose divisor is a multiple of \(-gp\). Mori and Matui reported that the complement of the set of all Weierstrass points is open dense. In this paper we shall count the number \( W(R) \) of the analytic arcs of the Weierstrass points.
2. We first prove the Theorem 1. Let $\mathbf{C}$ be the complex plane and define an injection $J$ by

$$U \ni z = x + iy \mapsto (x + i\theta, y + i\theta) \in \mathbf{C} \times \mathbf{C},$$

where $U \subseteq \{z | \vert z \vert < 1\}$, then there exists the unique analytic function $F = F(\tau, \omega)$ of a neighborhood $D$ of $J(U) \subseteq \mathbf{C} \times \mathbf{C}$ satisfying $f = F \circ J$. Let $f(p) = 0$ and $J(p) = (0, 0)$, then $F(0, 0) = 0$ and $F \not\equiv 0$ on $D$. Hereafter we may often replace $U$ and $D$ by some suitably small neighborhoods of $p$ and $(0, 0) \in \mathbf{C} \times \mathbf{C}$ respectively, but without notice we use the same notation for the sake of brevity.

By the Weierstrass' preparation theorem $F$ has the form as follows.

$$F(\tau, \omega) = \tau^m H(\tau, \omega) G(\tau, \omega),$$

which satisfies

1. $m$ is a non-negative integer,
2. $H(\tau, \omega)$ is a Weierstrass polynomial,
3. $G(\tau, \omega)$ is an analytic function of $D$ such that $G(\tau, \omega) \not\equiv 0$

for all $(\tau, \omega) \in D$.

When no confusion will arise we identify $U$ with $J(U)$. It follows that $U \cap \{(\tau, \omega) | \tau = 0\}$ is the $y$-axis of $U$. According to the unique factorization theorem we can suppose that $H(\tau, \omega)$ is irreducible. Hence the 1-dimensional irreducible analytic set $K = \{(\tau, \omega) | H(\tau, \omega) = 0\}$ in $D \subseteq \mathbf{C} \times \mathbf{C}$ can be described as follows by the uniformization theorem \[3, p.175\].

$$K = \{(\tau, \omega) | \tau = \alpha^s \text{ and } \omega = \sum_{n=1}^{\infty} c_n \alpha^n\},$$

where $s$ is a positive integer and $\omega = \omega(\alpha)$ is an analytic function of $\alpha$ on a suitable neighborhood of $\theta \in \mathbf{C}$. Moreover the mapping $\alpha \mapsto (\tau(\alpha), \omega(\alpha))$ is one-to-one, which is of use in Section 3.

If we set $\alpha = re^{i\theta}$ and $c_n = a_n + i b_n$ as usual, then we obtain

$$U \cap K = \{(x, y) | x = r^s \cos s \theta, \ y = \sum_{n=1}^{\infty} (a_n \cos n \theta - b_n \sin n \theta) r^n, \ r^s \sin s \theta = 0 \text{ and } \sum_{n=1}^{\infty} (a_n \sin n \theta + b_n \cos n \theta) r^n = 0\}.$$

Consider the case $r = 0$, then $\{(x, y) \in U \cap K | r = 0\} = \{(0, 0)\}$. 
While in the case \( r=0 \) we have the equation \( \sin s\theta = 0 \), which has the following 2\( s \) solutions.

\[ \theta_j = j\pi/s \text{ and } \sigma_j = \theta_j + \pi \ (j=0, 1, \ldots, s-1). \]

After this we fix \( \theta_j \). By the fact that

\[ |a_n \sin n\theta_j + b_n \cos n\theta_j| \leq 2|c_n|, \]

the following function \( g(r) \) is analytic in a neighborhood of 0 with respect to the real variable \( r \).

\[ g(r) = \sum_{n=1}^{\infty} (a_n \sin n\theta_j + b_n \cos n\theta_j) r^n. \]

If \( g(r) \equiv 0 \), then we can assume that \( g(r) \equiv 0 \) iff \( r=0 \) on its suitably small domain. We turn to the case \( g(r) \equiv 0 \), that is,

\[ a_n \sin n\theta_j + b_n \cos n\theta_j = 0 \]

for every \( n \). A similar argument yields that

\[ h(r) = \sum_{n=1}^{\infty} (a_n \cos n\sigma_j - b_n \sin n\sigma_j) r^n \]

is analytic on a neighborhood of 0 with respect to the real variable \( r \). Hence the arc \( z(r) = r^s \cos s\theta_j + i h(r) \ (r>0) \) is analytic and contained in \( U \cap K \). On the other hand it is easy to see that \( \cos s\sigma_j = (-1)^s \cos s\theta_j \) and

\[ a_n \sin n\sigma_j + b_n \cos n\sigma_j = 0 \]

for every \( n \). Moreover one computes

\[ \sum_{n=1}^{\infty} (a_n \cos n\sigma_j - b_n \sin n\sigma_j) r^n = \sum_{n=1}^{\infty} (a_n \cos n\theta_j - b_n \sin n\theta_j)(-1)^n r^n = h(-r). \]

Thus the analytic arc \( z(r) = (-r)^s \cos s\theta_j + i h(-r) \ (r>0) \) belongs to \( U \cap K \). This guarantees that the arc \( z(r) = r^s \cos s\theta_j + i h(r) \) belonging to \( U \cap K \) is analytic in a neighborhood of 0 with respect to the real variable \( r \). Therefore \( U \cap K \) is a sum of at most \( s \) analytic arcs, and so our result follows.

3. As a sharper form we may conclude that precisely \( U \cup K \) is an analytic arc when \( m=0 \) and \( H(\tau, \omega) \) is irreducible. Let \( \{n|c_n=0\} = \{u_k|k=1, 2, \ldots\} \). Suppose that there exists a common
divisor $q \geq 2$ of $\{s\} \cup \{u_k | k=1, 2, \ldots\}$, then one can write $s=qv$ and $u_k=qv_k$ ($k=1, 2, \ldots$) where $v$ and $v_k$ ($k=1, 2, \ldots$) are positive integers. Therefore the uniformization is represented as

$$\alpha \rightarrow ((a^0)^v, \sum_k c_u_k (a^0)^w k),$$

which contradicts the fact that the mapping is one-to-one. Consequently there exists an integer $t$ such that the greatest common divisor of $\{s\} \cup \{u_k | k \leq t\}$ is 1. We consider the simultaneous equations

$$\begin{align*}
\sin s \theta &= 0, \\
{a_u}_k \sin u_k \theta + {b_v}_k \cos u_k \theta &= 0 \quad (k=1, 2, \ldots, t).
\end{align*}$$

If $\mu$ is a solution, then $\mu+\pi$ is also a solution. Thus we can assume $0 \leq \mu < \pi$. It remains only to show that there is at most one solution. By a calculation, since $a^2_u + b^2_v \neq 0$, it suffices to prove

$$\begin{align*}
\sin s \theta &= 0 \\
\sin u_k \theta &= 0 \quad (k=1, 2, \ldots, t)
\end{align*}$$

has $\mu=0$ as its unique solution. If there exists a solution $\nu$ such that $0 < \nu < \pi$, then $\nu$ is written as follows.

$$\nu = d\pi / s = d_1 \pi / u_1 = \ldots = d_t \pi / u_t,$$

where $d$ and $d_k$ are positive integers such that $0 < d < s$ and $1 < d_k < u_k$ ($k=1, 2, \ldots, t$). Put $d/s = c/e$ where $c$ and $e$ are relatively prime numbers. Obviously $e$ is a divisor of $s$. Similarly $e$ is a divisor of $n_k$ for every $k$. This shows $e=1$, which is impossible.

Notice that $U \cap K - \{p\}$ is composed of at most two arcs which are analytic curves in the sense of Ahlfors and Sario [1, p.117].

4. In terms of a local coordinate system $z=x+y$ of an open disk $U \subset R$ the set of all Weierstrass points on $U$ can be written in the form $\{z | f(z)=0\}$, where

$$f(z) = | R^0 I^0 R^1 I^1 \ldots R^{e-1} I^{e-1}|$$

here

$$\begin{align*}
R^j &= (\text{Re } f_1^{(j)}(z), \text{ Re } f_2^{(j)}(z), \ldots, \text{ Re } f_{2g}^{(j)}(z)), \\
I^j &= (\text{Im } f_1^{(j)}(z), \text{ Im } f_2^{(j)}(z), \ldots, \text{ Im } f_{2g}^{(j)}(z))
\end{align*}$$
(j=0, 1, \ldots, g-1) and the functions f_k(z) (k=1, 2, \ldots, 2g) are real linearly independent and analytic on U. Therefore we obtain f(z) \neq 0 on U. Due to Theorem 1 the following theorem can be verified directly.

THEOREM 2. For any point p on R there exists a neighborhood U of p such that the set of all Weierstrass points on U is a finite sum of analytic arcs.

5. Two non-constant analytic arcs \alpha and \beta on a canonical subregion D are said to be equivalent if and only if there exists an analytic arc \gamma on R such that \alpha \cup \beta \subset \gamma. For any canonical subregion D we denote by W(D) the number of the union of the equivalence classes of the non-constant analytic arcs and the constant arcs of the Weierstrass points on D, which is finite due to the theorem. While there exists a canonical exhaustion \{D_n | n \geq 1\} which satisfies D_n \subset D_{n+1} and \lim_{n \to \infty} D_n, we put W(R) = \lim_{n \to \infty} W(D_n).

It is easy to verify that W(R) is independent of the choice of canonical exhaustion.

When R is compact the classical result can be written as follows.

\[ 2(g+1) \leq W(R) \leq (g-1)g(g+1). \]

When \Gamma_{he} \cap \Gamma_{hsc} \subset \Gamma_{he}, that is to say if \Gamma \cong 0_{K,D}, Watanabe [12] proved that

\[ W(R) \leq (g-1)g(g+1). \]

If \Gamma \cong 0_{K,D}, then there exists a compact continuation \tilde{R} of R which is conformally unique. Moreover any rational function on R can be uniquely extended as a meromorphic function on \tilde{R}. A point \tilde{p} \in \tilde{R} - R is called a Weierstrass point of R on the ideal boundary of R if \tilde{p} is a Weierstrass point of \tilde{R}. This definition is conformally unique. In this extended sense Watanabe's result can be written as follows.
PROPOSITION. If \( \Gamma_{he} \cap \Gamma_{he}' \subset \Gamma_{he}' \), then
\[
2(g+1) \leq W(R) \leq (g-1)g(g+1).
\]

6. Let \( W \) be the set of all Weierstrass points on the interior \( R \) of a bordered Riemann surface \( \bar{R} \) of finite genus \( g \). In this section we assume that \( R - W \) has no simply connected component which is relatively compact on \( R \).

Let \( S \) be a compact surface of genus \( h \) and \( \{ \alpha_j \mid 1 \leq j \leq u \} \) a union of closed curves on \( S \). For a given \( p \in R \) a non-negative integer \( N(p) \) is defined as follows. When there exist an open disk \( V \ni p \) and a union of open Jordan arcs \( \{ \beta_j \mid 1 \leq j \leq v \} \) with the property that \( \beta_j \cap \beta_k = \{ p \} \) \( (j \neq k) \), \( \bigcup_{j=1}^{v} \beta_j = \bigcup_{j=1}^{u} \alpha_j \cap V \) and \( V - \bigcup_{j=1}^{v} \beta_j \) is composed of \( 2v \) simply connected components, put \( N(p) = v - 1 \). In the other case put \( N(p) = 0 \). Under this condition we have the following lemma.

**LEMMA.** If there exists a point \( q \in S \) such that \( \alpha_j \cap \alpha_k = \{ q \} \) \( (j \neq k) \) and \( u \geq 2h \), then the number of simply connected components of \( S - \bigcup_{j=1}^{u} \alpha_j \) is at least \( u - 2h + 1 \).

The lemma can be reduced to the case when the curves \( \alpha_j \) are closed Jordan curves. If \( h = 0 \), the result holds trivially. Due to contractions of closed curves to the point \( q \), the lemma can be proved by induction on \( h \).

If \( \sum_{p \in S} N(p) \geq 2h - 1 \), then by means of tying up with the components of \( \bigcup_{j=1}^{u} \alpha_j \) via Jordan arcs on \( S - \bigcup_{j=1}^{u} \alpha_j \) with end points on \( \bigcup_{j=1}^{u} \alpha_j \) the situation resolves itself into the pattern of the lemma by the usual method of topological deformations, and an easy calculation shows that \( S - \bigcup_{j=1}^{u} \alpha_j \) has a simply connected component.
If $\bar{R}$ is compact, then the genus of the double $\hat{R}$ of $R$ is $2g+m-1$, where $m$ stands for the number of contours of $\bar{R}$. Let $W^*$ be the symmetric set of $W$. Then $W \cup (\bar{R} - R) \cup W^*$ in $\hat{R}$ is composed of a finite number of closed curves $\{a_j | 1 \leq j \leq u\} = W \cup (\bar{R} - R) \cup W^*$, which has the local property indicated in the theorem. Therefore our assumption guarantees the following inequality.

$$\sum_{p \in \hat{R}} N(p) \leq 4g + 2m - 4.$$

If $\bar{R}$ is not necessarily compact with $m$ contours and the double $\hat{R}$ is of class $O_{K,D}$ [cf. 8, Lemma 3], then we can extend the concept of the Weierstrass points on the ideal boundary in a similar manner as in the preceding section 3, and again we have

$$\sum_{p \in \hat{R}} N(p) \leq 4g + 2m - 4$$

as above under the similar assumption.

REFERENCES

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CHUKYO UNIVERSITY, NAGOYA, JAPAN